2.3: MODELING WITH FIRST ORDER ODES

Review

- If f is **proportional to** g, then it means

$f = kg$, where k is the "constant of proportionality"

Exercise 1

Suppose we initially have 3 rabbits. After 2 years, we have 14 rabbits. Assuming the population growth of the rabbits is proportional to the number of rabbits, how many rabbits will we have in 3 more years?

$R(t) = \#$ of rabbits

$t =$ time (years)

$$\frac{dR}{dt} = kR, \quad R(0) = 3, \quad R(2) = 14$$

What is $R(5)$?

Solve diff eq:

$$\int \frac{dR}{R} = \int k\,dt$$

$$\ln|R| = kt + c$$

$$R(t) = e^{kt+c} = e^{kt}e^{c} = ce^{kt}$$

Solve for c and k:

$$R(0) = ce^{k\cdot 0} = \underline{c = 3}$$

$$R(2) = 3e^{k\cdot 2} = 14$$

$$\Rightarrow e^{2k} = \frac{14}{3}$$

$$\Rightarrow 2k = \ln\left(\frac{14}{3}\right)$$

$$\Rightarrow k = \frac{1}{2}\ln\left(\frac{14}{3}\right)$$

$$R(t) = 3e^{\frac{1}{2}\ln\left(\frac{14}{3}\right)t}$$

$$\boxed{R(5) = 3e^{\frac{1}{2}\ln\left(\frac{14}{3}\right)\cdot 5} \text{ rabbits}}$$

$$\boxed{\approx 141 \text{ rabbits}}$$

Exercise 2

annual

Suppose we invest \$100 per month in a savings account that makes 5% interest compounded continuously. Initially, we have nothing in the savings account. How much money will be in the account after 8 years?

$S(t)$ = dollars in the account

t = time (years)

$$\frac{dS}{dt} = +1200 + 0.05 \cdot S(t), \quad S(0) = 0$$

$$S' - 0.05 \cdot S = 1200$$

$$\mu S' \underbrace{- 0.05\mu}_{\frac{d\mu}{dt}} \cdot S = 1200\mu$$

$$\frac{d\mu}{dt} = -0.05\mu \Rightarrow \mu(t) = e^{-0.05t}$$

$$\frac{d}{dt}\left(e^{-0.05t} \cdot S(t)\right) = 1200e^{-0.05t}$$

$$e^{-0.05t} \cdot S(t) = \frac{1200}{-0.05} e^{-0.05t} + C$$

$$S(t) = \frac{1200}{-0.05} + Ce^{0.05t}$$

$$S(0) = \frac{1200}{-0.05} + c\, e^{0.05(0)} = \frac{1200}{-0.05} + c = 0$$

$$C = \frac{1200}{0.05}$$

$$S(t) = \frac{-1200}{0.05} + \frac{1200}{0.05} e^{0.05t}$$

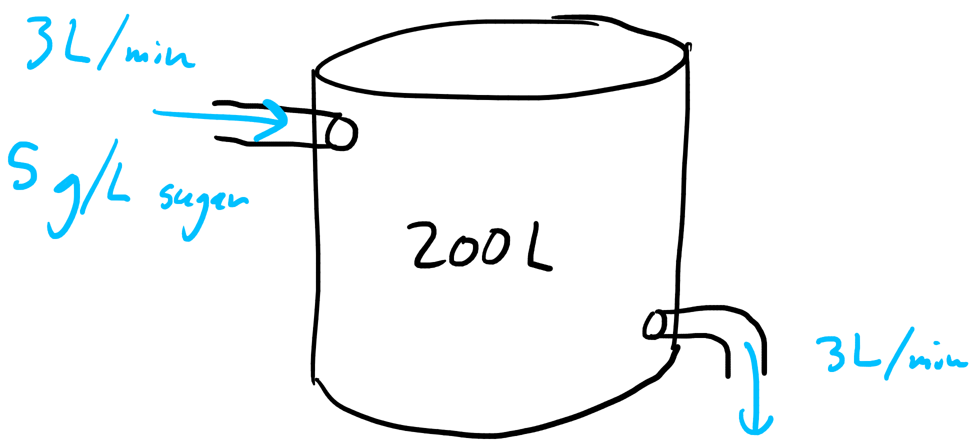
$$\boxed{S(8) = \frac{-1200}{0.05} + \frac{1200}{0.05} e^{0.05(8)} \text{ dollars}}$$

Exercise 3

Suppose we have a 200 L tank filled with water. We start pouring sugar water into the tank at a rate of 3 L/min. The sugar water contains 5 g/L of sugar. At the same time, the well-mixed fluid flows out of the tank at a rate of 3 L/min. How much sugar is in the tank after 1 hr?

$S(t)$ = grams of sugar in the tank

t = time (minutes)



$$\frac{dS}{dt} = +\underbrace{\left(3\frac{L}{min}\right)\left(5\frac{g}{L}\right)}_{\text{sugar flowing in}} - \underbrace{\left(3\frac{L}{min}\right)\left(\frac{S(t)\,g}{200\,L}\right)}_{\text{sugar flowing out}}, \quad S(0) = 0.$$

$$\frac{dS}{dt} = 15 - \frac{3}{200}S$$

$$\int \frac{dS}{15 - \frac{3}{200}S} = \int dt$$

$$-\frac{200}{3}\ln\left|15 - \frac{3}{200}S\right| = t + c$$

$$\ln\left|15 - \frac{3}{200}S\right| = \frac{-3}{200}t + c$$

$$15 - \frac{3}{200}S = e^{\frac{-3t}{200}+c} = e^{c}e^{\frac{-3t}{200}} = ce^{\frac{-3t}{200}}$$

$$\frac{-3}{200}S(t) = ce^{\frac{-3t}{200}} - 15$$

$$S(t) = ce^{\frac{-3t}{200}} + \frac{15 \cdot 200}{3} = ce^{\frac{-3t}{200}} + 1000$$

$$S(0) = ce^{0} + 1000 = 0 \Rightarrow c = -1000$$

$$S(t) = -1000e^{\frac{-3t}{200}} + 1000$$

$$\boxed{S(60) = -1000e^{\frac{-180}{200}} + 1000 \text{ grams}}$$

Exercise 4

Suppose we leave a bucket of ice cream out on the counter in a 70 °F room. The ice cream was initially 15 °F. However, after 1 minute of sitting there, it's temperature is 20 °F. How long until the ice cream starts to melt? (Assume that the ice cream obeys Newton's law of cooling: The rate at which the temperature changes is proportional to the temperature difference of the object and its surroundings.)

$T(t)$ = temp of ice cream (°F)

t = time (minutes)

$$\frac{dT}{dt} = k(T(t) - 70), \quad T(0) = 15, \quad T(1) = 20$$

When is $T(t) = 32$?

$$\int \frac{dt}{T-70} = \int k\,dt$$

$$\ln|T-70| = kt + c$$

$$T - 70 = e^{kt+c} = e^c e^{kt} = ce^{kt}$$

$$T(t) = 70 + ce^{kt}$$

$$T(0) = 70 + ce^{k(0)} = 70 + c = 15$$

$$\Rightarrow c = -55$$

$$T(1) = 70 - 55e^{k \cdot 1} = 20$$

$$\Rightarrow -55e^{k} = -50$$

$$\Rightarrow e^{k} = \frac{50}{55} = \frac{10}{11}$$

$$k = \ln\left(\frac{10}{11}\right)$$

$$T(t) = 70 - 55e^{\ln\left(\frac{10}{11}\right)t} = 32$$

$$-55e^{\ln\left(\frac{10}{11}\right)t} = -38$$

$$e^{\ln\left(\frac{10}{11}\right)t} = \frac{+38}{+55}$$

$$\ln\left(\frac{10}{11}\right)t = \ln\left(\frac{38}{55}\right)$$

$$\boxed{t = \frac{\ln\left(\frac{38}{55}\right)}{\ln\left(\frac{10}{11}\right)} \text{ min}}$$

2.4: EXISTENCE AND UNIQUENESS OF SOLUTIONS

Review

- A solution **exists** if

  there is at least one solution.

- A solution is **unique** if

  there is only one solution.

- **Theorem for linear ODEs:** If p and g are continuous on an interval $I = (a, b)$ containing the initial condition t_0, then the initial value problem

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0$$

  has a unique solution on I.

- **Theorem for nonlinear ODEs:** Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $(a, b) \times (c, d)$ containing the point (t_0, y_0). Then, there is a unique solution to the initial value problem

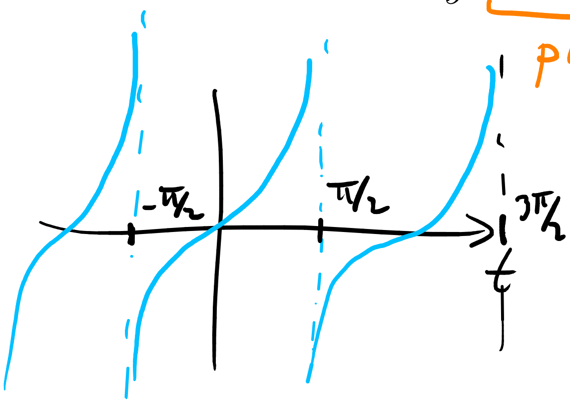
$$y' = f(t, y), \qquad y(t_0) = y_0$$

  on a sufficiently small interval $I_h = (t_0 - h, t_0 + h)$ around t_0.

Exercise 5

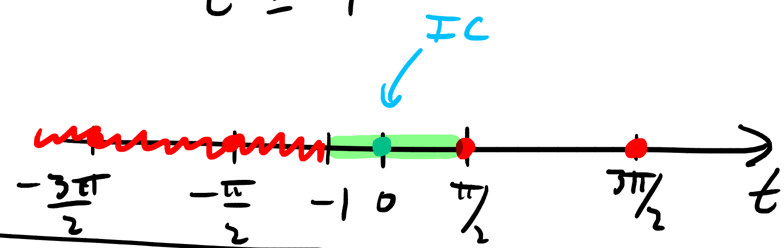
Without solving the IVP, determine where a unique solution is guaranteed to exist.

$$y' \underbrace{- \tan(t)}_{p(t)} y = \underbrace{\sqrt{t+1},}_{g(t)} \qquad y(0) = 2\pi.$$



$t + 1 \geq 0$

$t \geq -1$



$\boxed{\text{This is a unique solution on } (-1, \pi/2).}$

Exercise 6

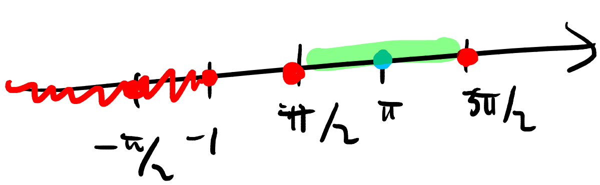
Without solving the IVP, determine where a unique solution is guaranteed to exist.

$$\cos(x) f' - 4x^2 f = \ln(1 + x), \qquad f(\pi) = 7.$$

$$f' - \underbrace{\frac{4x^2}{\cos(x)}}_{p(t)} f = \underbrace{\frac{\ln(1+x)}{\cos(x)}}_{g(t)}$$

$1 + x > 0$

$x > -1$



$\boxed{\text{There is a unique solution on } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).}$

Exercise 7

Consider the differential equation

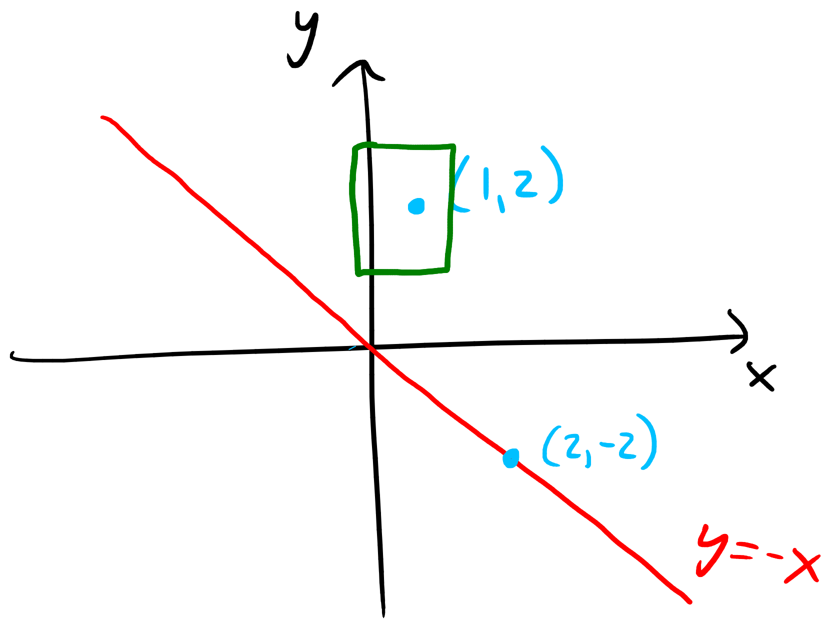
$$y' = (x+y)^{1/3}$$

If the initial condition is $y(2) = -2$, does the IVP have a unique solution? What if the initial condition is $y(1) = 2$?

$$f(x,y) = (x+y)^{1/3}$$

$$\frac{\partial f}{\partial y} = \frac{1}{3}(x+y)^{-2/3}$$

$$= \frac{1}{3}\,\frac{1}{(x+y)^{2/3}}$$



can't have $x+y=0$

$y=-x$

When the IC is $y(2)=-2$, we have no idea if there is a unique solution or not.

When the IC is $y(1)=2$, we have a unique solution on at least some small interval.

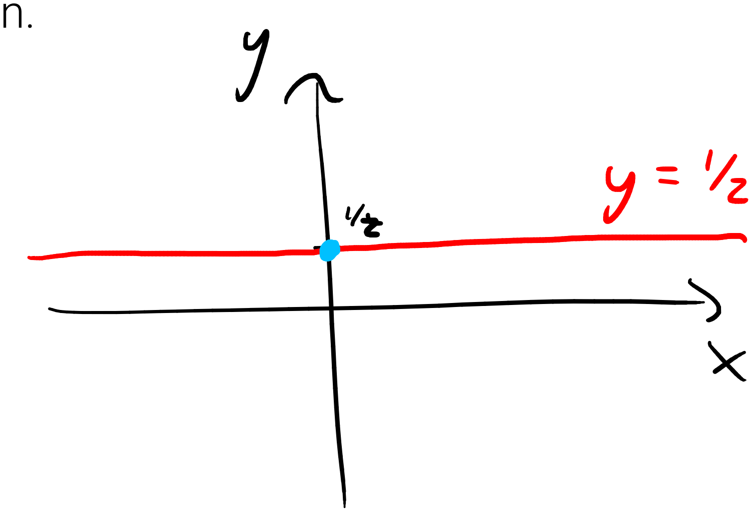
Exercise 8

Consider the initial value problem

$$y' = \frac{x}{\underbrace{2y-1}_{f}}, \qquad y(0) = \frac{1}{2}$$

Why does the existence and uniqueness theorem not apply to this IVP? Show that this IVP has more than one solution.

$$f(x,y) = \frac{x}{2y-1}$$



$$\frac{\partial f}{\partial y} = \frac{-2x}{(2y-1)^2}$$

Theorem does not apply because IC is at a point where f and $\frac{\partial f}{\partial y}$ are discontinuous.

$$\frac{dy}{dx} = \frac{x}{2y-1} \Rightarrow \int (2y-1)\,dy = \int x\,dx$$

$$y^2 - y = \frac{1}{2}x^2 + C$$

$$\left(\frac{1}{2}\right)^2 - \frac{1}{2} = \frac{1}{2}(0)^2 + C$$

$$\Rightarrow C = -\frac{1}{4}$$

$$y^2 - y = \frac{1}{2}x^2 - \frac{1}{4}$$

implicit form of solution

$$y^2 - y - \frac{1}{2}x^2 + \frac{1}{4} = 0$$

$a = 1$ $b = -1$ $c = -\frac{1}{2}x^2 + \frac{1}{4}$

$$y = \frac{1 \pm \sqrt{1 - 4(1)\left(-\frac{1}{2}x^2 + \frac{1}{4}\right)}}{2}$$

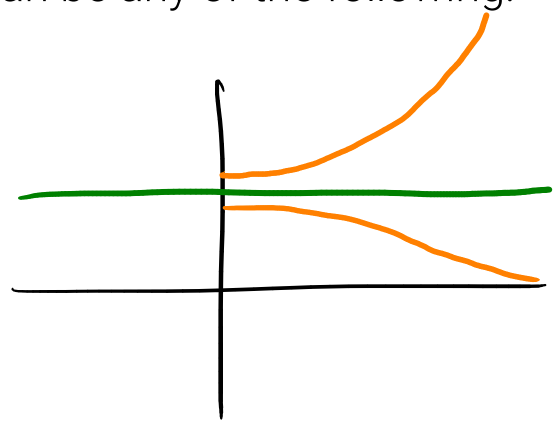
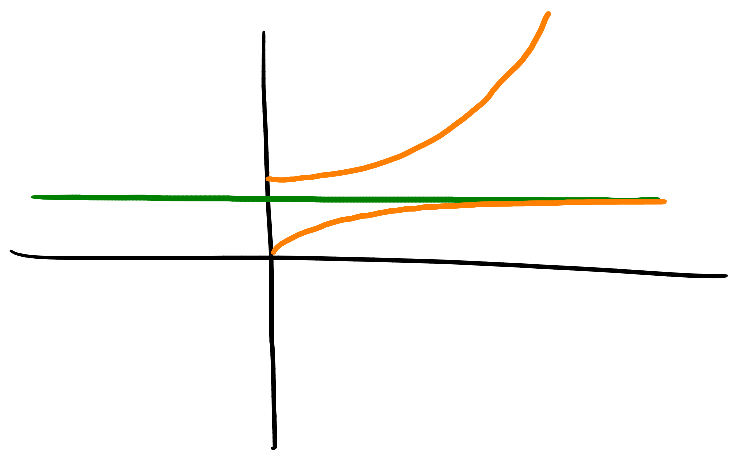
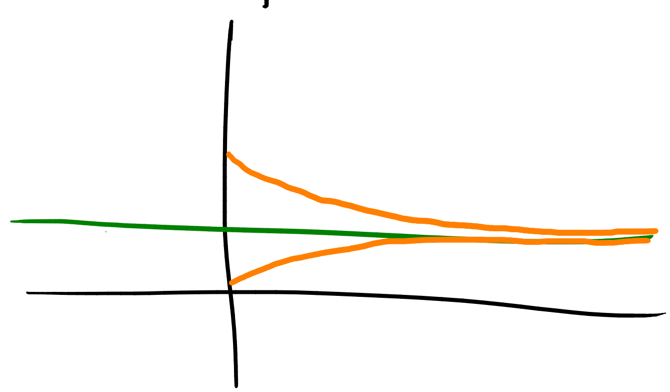
$$= \frac{1}{2} \pm \frac{\sqrt{1 + 2x^2 - 1}}{2}$$

$$\boxed{= \frac{1}{2} \pm \frac{\sqrt{2x^2}}{2}}$$

two solutions to the IVP

2.5: AUTONOMOUS EQUATIONS

Review

- A differential equation is **autonomous** if
  the independent variable doesn't appear "by itself".
- An **equilibrium solution** to a differential equation is
  a constant solution.
- The **stability** of an equilibrium solution can be any of the following:
  - **Unstable**
    If you start near it, you go away.
    
  - **Semi-stable**
    If you start near on one side, you go away. But if you start near on the other side, you go towards it.
    
  - **(Asymptotically) stable**
    If you start near it, you go towards it.
    

Exercise 9

Are the following autonomous or not?

1. $y'' + y' + y = 7$

   Yes

2. $x^2 - y' = y^4$

   No.

3. $f'(x) - 3f(x) - 12 = 0$ $\Rightarrow$ $\underline{f' - 3f - 12 = 0}$

   Yes.

4. $\dfrac{g''}{g'} + g = \sqrt{g}$

   Yes.

5. $\dfrac{w''(x)}{x^2 + 1} - (w(x))^{3/2} = 6\sin(x)$

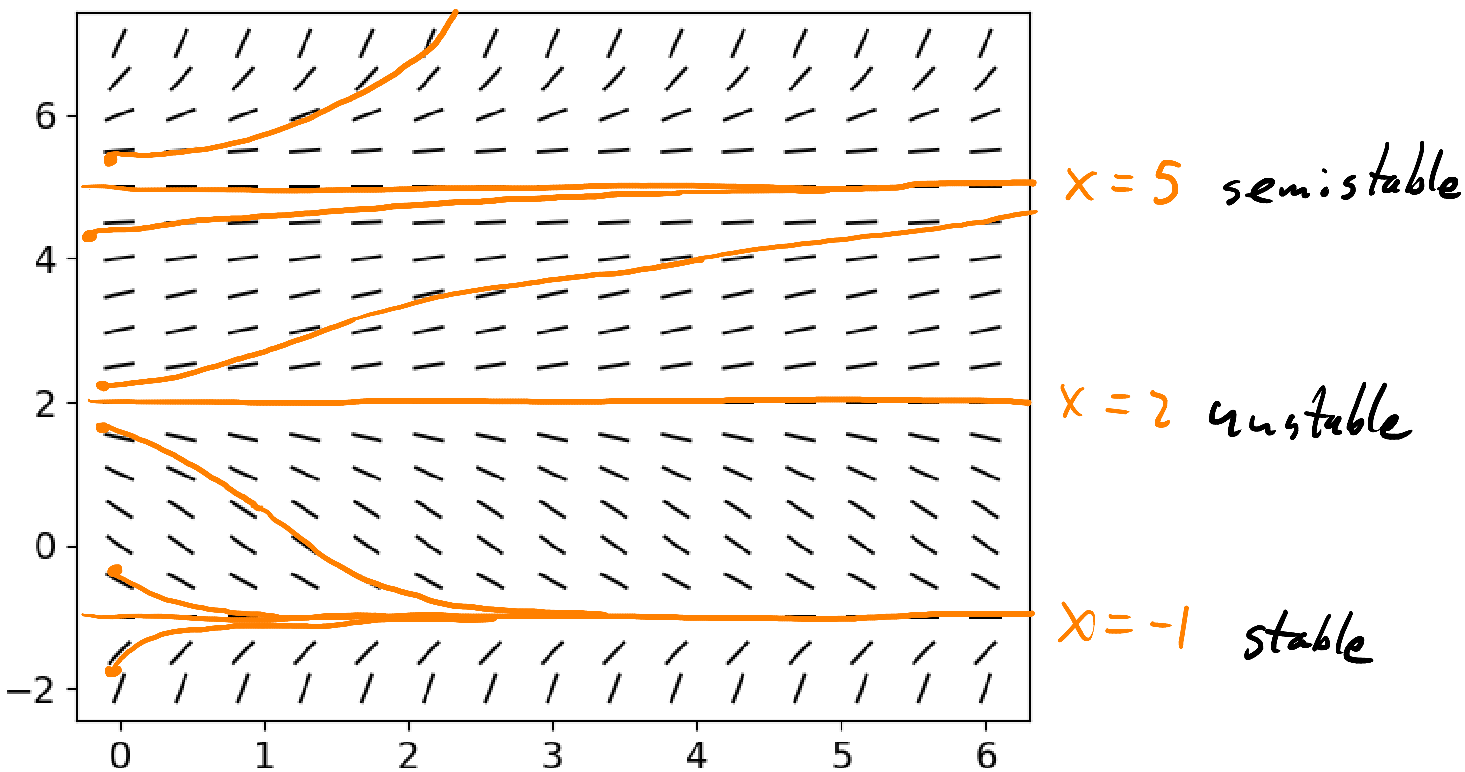
   No.

6. $\cos(u^2) + \dfrac{\mathrm{d}u}{\mathrm{d}x} = u$

   Yes.

Exercise 10

Given the following slope field, determine the equilibrium solutions and their stability. Also, draw the phase line diagram.



Exercise 11

Suppose the population of armadillos is governed by the equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 1000A - A^2.$$

What are the equilibrium solutions? Interpret them physically. What is the stability of each equilibrium solution? Interpret this physically.

To find the equilibrium solutions, set $\frac{dA}{dt} = 0$.

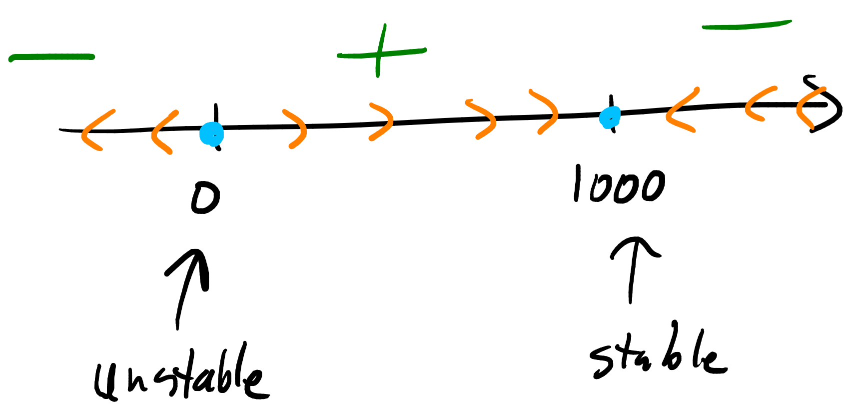
$$0 = 1000A - A^2$$

$$= A(1000 - A)$$

$$\boxed{A = 0 \quad \text{or} \quad A = 1000}$$

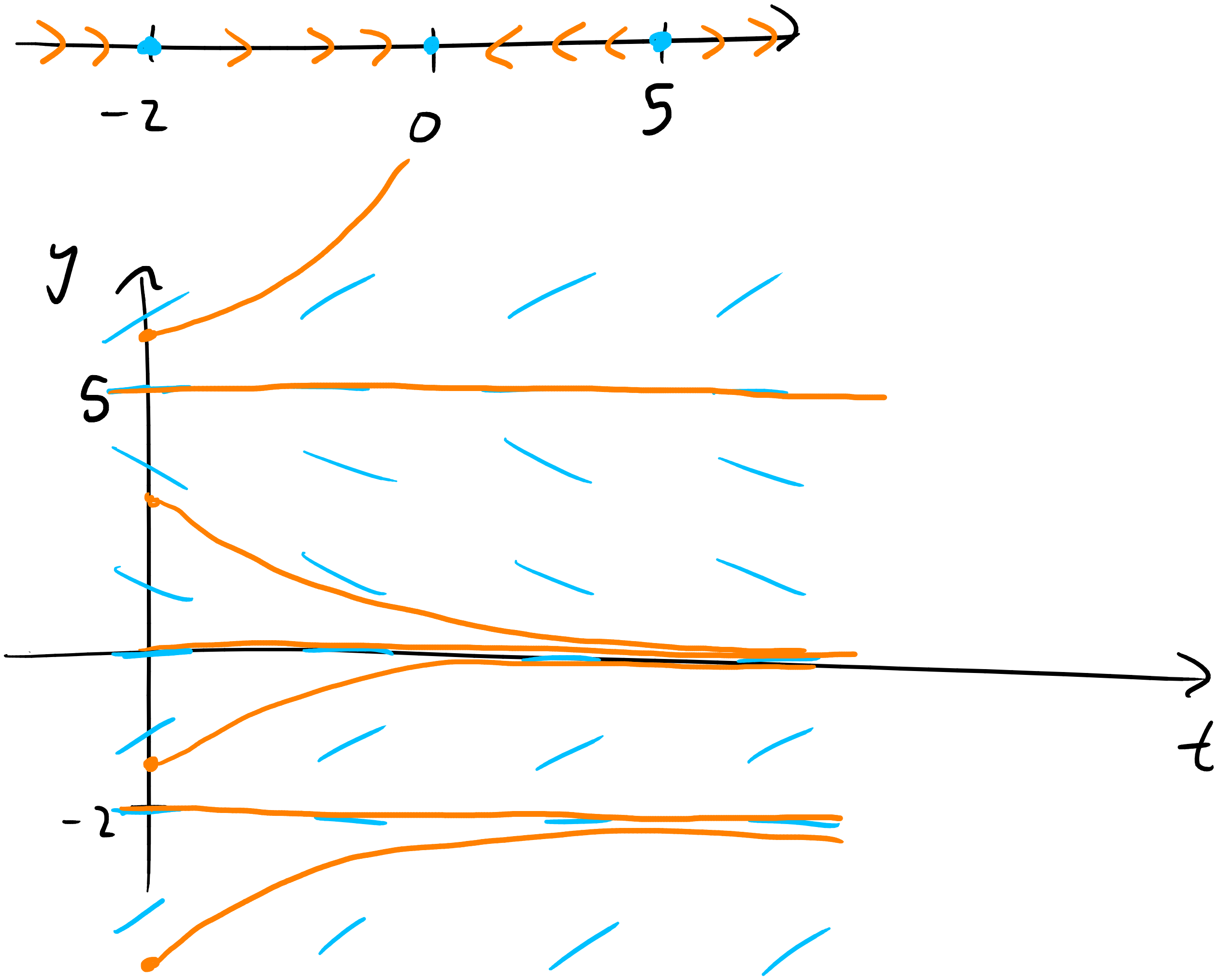
If we start with 0 armadillos, there will never be any armadillos.

If we start with 1000, there will always be 1000.



Exercise 12

Given the phase line diagram, sketch the corresponding slope field. Draw some representative solutions on the slope field for different initial conditions. Also, determine the stability of each equilibrium point.



-2 is semistable

0 is stable

5 is unstable.