SESSION 4: SECTIONS 2-3 AND 2-4

Introductory Derivative Rules

Constant:	$\frac{d}{dx}(k) = 0$	where k is any real number
Power:	$\frac{d}{dx}(x^n) = nx^{n-1}$	where n is any real number
Special Case:	$\frac{d}{dx}(x) = 1$	(because $x = x^1$)
Exponential:	$\frac{d}{dx}(b^x) = b^x \ln(b)$	where b is any positive real number
Special Case:	$\frac{d}{dx}(e^x) = e^x$	(because $\ln(e) = 1$)
Logarithm:	$\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)}$	where b is any positive real number
Special Case:	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	(because $\ln(e) = 1$)
Sum/Difference:	$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$	where f and g are differentiable functions
Constant Multiple:	$\frac{d}{dx}(k \cdot f(x)) = k\left(\frac{d}{dx}(f(x))\right)$	where k is any real number and f is a differentiable function

1. Find $f'(t)$ given $f(t) = 2t^2 - 3t + 1$.

$$f'(t) = 2\frac{d}{dt}(t^2) - 3\frac{d}{dt}(t) + \frac{d}{dt}(1)$$
$$= 2(2t) - 3(1) + 0$$
$$= \boxed{4t - 3}$$

2. Given $y = \frac{5}{9t^6} + 6\sqrt[3]{t^2}$, find $\frac{dy}{dt}$.

$$y = \frac{5}{9}t^{-6} + 6t^{2/3}$$
$$\frac{dy}{dt} = \frac{5}{9}\frac{d}{dt}(t^{-6}) + 6\frac{d}{dt}(t^{2/3})$$
$$= \frac{5}{9}(-6t^{-7}) + 6\left(\frac{2}{3}t^{-1/3}\right)$$
$$= -\frac{30}{9}t^{-7} + \frac{12}{3}t^{-1/3} = \boxed{-\frac{10}{3}t^{-7} + 4t^{-1/3}}$$

3. Find $\dfrac{d}{dx}\left(\pi x^{2\pi} + \dfrac{5x^8}{\sqrt{x}} + \dfrac{3e}{\sqrt[6]{x^5}}\right)$.

$$\frac{d}{dx}\left(\pi x^{2\pi}\right) + \frac{d}{dx}\left(5x^8 \cdot x^{-1/2}\right) + \frac{d}{dx}\left(3e\,x^{-5/6}\right)$$

$$= \pi \frac{d}{dx}\left(x^{2\pi}\right) + 5\frac{d}{dx}\left(x^{15/2}\right) + 3e\frac{d}{dx}\left(x^{-5/6}\right)$$

$$= \pi\left(2\pi x^{2\pi-1}\right) + 5\left(\frac{15}{2}x^{13/2}\right) + 3e\left(-\frac{5}{6}x^{-11/6}\right)$$

$$\boxed{= 2\pi^2 x^{2\pi-1} + \frac{75}{2}x^{13/2} - \frac{15e}{6}x^{-11/6}}$$

4. Given $f(x) = 3e^x + 4\ln(x) - \dfrac{1}{2}\log_7(x)$, find $f'(x)$.

$$f'(x) = 3\frac{d}{dx}\left(e^x\right) + 4\frac{d}{dx}\left(\ln x\right) - \frac{1}{2}\frac{d}{dx}\left(\log_7(x)\right)$$

$$= 3e^x \underbrace{\ln e}_{=1} + 4\left(\frac{1}{x\underbrace{\ln e}_{=1}}\right) - \frac{1}{2}\left(\frac{1}{x\ln 7}\right)$$

$$\boxed{= 3e^x + \frac{4}{x} - \frac{1}{2x\ln 7}}$$

5. Find $\dfrac{dp}{dx}$ given $p = 10^x + x^7 + \underbrace{\log(x^5)}_{5\log x} + 10e^x$.

$$\frac{dp}{dx} = \frac{d}{dx}\left(10^x\right) + \frac{d}{dx}\left(x^7\right) + 5\frac{d}{dx}\left(\log x\right) + 10\frac{d}{dx}\left(e^x\right)$$

$$= 10^x \ln 10 + 7x^6 + 5\left(\frac{1}{x\ln 10}\right) + 10e^x \underbrace{\ln e}_{=1}$$

$$\boxed{= 10^x \ln 10 + 7x^6 + \frac{5}{x\ln 10} + 10e^x}$$

6. Find $f'(x)$ given $f(x) = \ln\left(\frac{x^2}{5}\right) + \log(2x)$.

$$f(x) = \ln(x^2) - \ln 5 + \log 2 + \log x$$
$$= 2\ln(x) - \ln 5 + \log 2 + \log x$$
$$f'(x) = 2\frac{d}{dx}(\ln x) - \frac{d}{dx}(\ln 5) + \frac{d}{dx}(\log 2) + \frac{d}{dx}(\log x)$$
$$= 2\left(\frac{1}{x}\right) - 0 + 0 + \frac{1}{x\ln 10}$$
$$= \boxed{\frac{2}{x} + \frac{1}{x\ln 10}}$$

7. If $h(x) = -4f(x) + 5g(x) - 9$, $f'(5) = 8$, and $g'(5) = 4$, find $h'(5)$.

$$h'(x) = -4f'(x) + 5g'(x) - 0$$
$$h'(5) = -4f'(5) + 5g'(5) - 0$$
$$= -4(8) + 5(4)$$
$$= -32 + 20$$
$$\boxed{= -12}$$

8. Given $f(x)$ is a polynomial function such that $\underbrace{f(2) = 17}_{\text{goes through } (2,17)}$ and $f'(x) = 12x^3 - 12x$, find the equation of the tangent line at $x = 2$.

$$f'(2) = 12(2)^3 - 12(2)$$
$$= 12(8) - 24$$
$$= 96 - 24$$
$$= 72 \leftarrow \text{slope of tangent}$$

$$y - y_1 = m(x - x_1)$$
$$y - 17 = 72(x - 2)$$
$$y = 72x - 144 + 17$$
$$= 72x - 127$$

Estimating the Cost/Revenue/Profit of a Single Item Using Marginal Analysis

- The approximate cost of making the n^{th} item is $C'(n-1)$.
- The approximate revenue from selling the n^{th} item is $R'(n-1)$.
- The approximate profit from making and selling the n^{th} item is $P'(n-1)$.

The Exact Cost/Revenue/Profit of a Single Item

- The exact cost of making the n^{th} item is $C(n) - C(n-1)$.
- The exact revenue from selling the n^{th} item is $R(n) - R(n-1)$.
- The exact profit from making and selling the n^{th} item is $P(n) - P(n-1)$.

9. The total profit (in dollars) from the sale of x skateboards is $P(x) = 30x - 0.3x^2 - 250$ for $0 \le x \le 100$.

   (a) Find the $\underbrace{\text{exact profit from the sale of the } 26^{th}}_{P(26)-P(25)}$ skateboard.

   $P(26) = 327.2$

   $P(25) = 312.5$

   $P(26) - P(25) = 327.2 - 312.5$

   $= 14.7$

   The company will profit by \$14.70 when the 26th skateboard is sold.

   (b) Find the $\underbrace{\text{marginal profit function}}_{P'(x)}$ and then use it to $\underbrace{\text{approximate the profit from the sale of the } 26^{th}}_{P'(25)}$ skateboard.

   $P'(x) = 30 - .6x$

   $P'(25) = 30 - .6(25) = 15$

   The approximate profit from selling the 26th skateboard is \$15.

For the rules below, assume f and g are differentiable functions.

The Product Rule:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\left(\frac{d}{dx}\left(g(x)\right)\right) + g(x)\left(\frac{d}{dx}\left(f(x)\right)\right)$$

$$= f(x)\cdot g\,'(x) + g(x)\cdot f\,'(x)$$

The Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\left(\frac{d}{dx}\left(f(x)\right)\right) - f(x)\left(\frac{d}{dx}\left(g(x)\right)\right)}{\left(g(x)\right)^2}$$

$$= \frac{g(x)\cdot f\,'(x) - f(x)\cdot g\,'(x)}{\left(g(x)\right)^2}$$

10. Find $f'(x)$ for the following functions. You do not need to simplify the functions after applying the derivative rules.

(a) $f(x) = x^2\ln(x)$

$$f'(x) = \frac{d}{dx}(x^2)\cdot\ln(x) + x^2\frac{d}{dx}(\ln x)$$

$$= 2x\ln(x) + x^2\left(\frac{1}{x\underbrace{\ln e}_{=1}}\right)$$

$$= 2x\ln(x) + x$$

(b) $f(x) = \dfrac{4e^x}{7x^3+2x^2+5x}$

$$f'(x) = \frac{(7x^3+2x^2+5x)\frac{d}{dx}(4e^x) - 4e^x\frac{d}{dx}(7x^3+2x^2+5x)}{(7x^3+2x^2+5x)^2}$$

$$= \frac{(7x^3+2x^2+5x)(4e^x\overbrace{\ln e}^{1}) - 4e^x(21x^2+4x+5)}{(7x^3+2x^2+5x)^2}$$

$$= \frac{(7x^3+2x^2+5x)(4e^x) - 4e^x(21x^2+4x+5)}{(7x^3+2x^2+5x)^2}$$

(c) $f(x) = \dfrac{\log_3(x^6)}{e^2 + \sqrt[5]{x^3}} = \dfrac{6\log_3 x}{e^2 + x^{3/5}}$

$$f'(x) = \frac{(e^2 + x^{3/5})\frac{d}{dx}(6\log_3 x) - (6\log_3 x)\frac{d}{dx}(e^2 + x^{3/5})}{(e^2 + x^{3/5})^2}$$

$$= \frac{(e^2 + x^{3/5})(6 \cdot \frac{1}{x\ln 3}) - 6\log_3 x\,(\frac{3}{5}x^{-2/5})}{(e^2 + x^{3/5})^2}$$

(d) $f(x) = \dfrac{5^x}{\sqrt[3]{x^2}} = \dfrac{5^x}{x^{2/3}} = 5^x \cdot x^{-2/3}$

work using quotient rule

$$f'(x) = \frac{x^{2/3}\frac{d}{dx}(5^x) - 5^x\frac{d}{dx}(x^{1/3})^2}{(x^{2/3})^2}$$

$$\boxed{= \frac{x^{2/3} \cdot 5^x \cdot \ln 5 - 5^x(\frac{2}{3}x^{-1/3})}{x^{4/3}}}$$

If we simplify we get

$$\frac{5^x x^{-1/3}(x\ln 5 - \frac{2}{3})}{x^{4/3}} = \frac{5^x(x\ln 5 - \frac{2}{3})}{x^{5/3}}$$

use product rule

$$f'(x) = 5^x\frac{d}{dx}(x^{-2/3}) + \frac{d}{dx}(5^x)(x^{-2/3})$$

$$\boxed{= 5^x(-\tfrac{2}{3}x^{-5/3}) + 5^x\ln 5\,(x^{-2/3})}$$

If we simplify

$$x^{-5/3}\,5^x(-\tfrac{2}{3} + x\ln 5)$$

$$= \frac{5^x(-\frac{2}{3} + x\ln 5)}{x^{5/3}}$$

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11. Find the x-value(s) where the graph of $f(x) = e^x(x^2 - 2x - 2)$ has $\underbrace{\text{a horizontal tangent line}}_{f'(x)=0}$.

(Note: In the interest of time, I'll provide this: $f'(x) = e^x(x^2 - 4)$. But you should use derivative rules to ensure you could find this derivative correctly. If you get stuck come to a math learning center help session or visit your instructor's office hours.)

$0 = e^x(x^2-4)$

~~$e^x = 0$~~ This is never zero

$x^2 - 4 = 0$

$\sqrt{x^2} = \sqrt{4}$

$\boxed{x = \pm 2}$

To find derivative use product rule:

$f'(x) = e^x \frac{d}{dx}(x^2-2x-2) + \frac{d}{dx}(e^x)(x^2-2x-2)$

$= e^x(2x-2) + e^x(x^2-2x-2)$

$= e^x(\cancel{2x} - 2 + x^2 - \cancel{2x} - 2)$

$= e^x(x^2-4)$

12. Find the x-value(s) where $f(x) = \dfrac{-20x}{x+2}$ has an $\underbrace{\text{instantaneous rate of change of } -10}_{f'(x)=-10}$.

$f'(x) = \dfrac{(x+2)\frac{d}{dx}(-20x) - (-20x)\frac{d}{dx}(x+2)}{(x+2)^2}$

$= \dfrac{(x+2)(-20) - (-20x)(1)}{(x+2)^2}$

$= \dfrac{-20x - 40 + 20x}{(x+2)^2}$

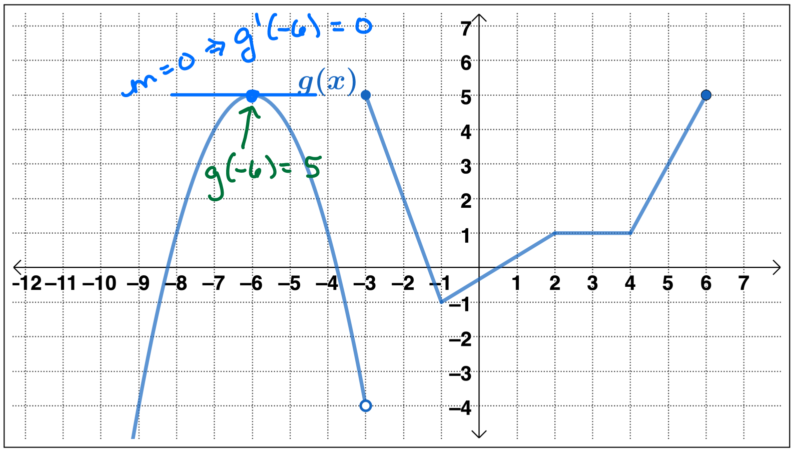
$-10 = \dfrac{-40}{(x+2)^2}$

$(x+2)^2 = 4$

$x+2 = 2$ $\quad$ $x+2 = -2$

$\boxed{x = 0}$ $\quad$ $\boxed{x = -4}$

13. Use the table and graph below to find each of the following.



x	$f(x)$	$f'(x)$
-6	4	10
-5	2	9
-4	0	10
-3	-4	6
-2	9	0
-1	3	-1
0	9	-4
1	8	0
2	7	4
3	3	5
4	1	7
5	-1	3

(a) $h'(-6)$ if $h(x) = x^2 g(x)$

$$h'(x) = x^2 \frac{d}{dx}(g(x)) + \frac{d}{dx}(x^2)\, g(x)$$
$$= x^2 \cdot g'(x) + 2x\, g(x)$$
$$h'(-6) = (-6)^2 \cdot \underbrace{g'(-6)}_{0} + 2(-6)\underbrace{g(-6)}_{5}$$
$$= 36(0) - 12(5)$$
$$= \boxed{-60}$$

(b) $p'(0)$ if $p(x) = \dfrac{e^x g(x)}{f(x)}$.

(*Hint*: The line segment on the interval $(-1, 2)$ contains the points $(-1, -1)$ and $(2, 1)$. The line that contains these two points is $y = \frac{2}{3}x - \frac{1}{3}$. Be sure you can calculate the equation of this line on your own and use this to help solve this problem.)

$$p'(x) = \frac{f(x)\frac{d}{dx}(e^x g(x)) - e^x g(x)\frac{d}{dx}(f(x))}{[f(x)]^2}$$
$$= \frac{f(x)\big(\overbrace{e^x g'(x) + e^x g(x)}^{\text{product rule}}\big) - e^x g(x) f'(x)}{[f(x)]^2}$$
$$p'(0) = \frac{f(0)\left(e^0 g'(0) + e^0 g(0)\right) - e^0 g(0) f'(0)}{[f(0)]^2} = \frac{9\left(1 \cdot \frac{2}{3} + 1 \cdot -\frac{1}{3}\right) - \left(-\frac{1}{3}\right)(-4)}{9^2}$$
$$= \frac{9\left(\frac{1}{3}\right) - \frac{4}{3}}{81}$$
$$= \frac{5/3}{81}$$
$$\boxed{= 5/243}$$