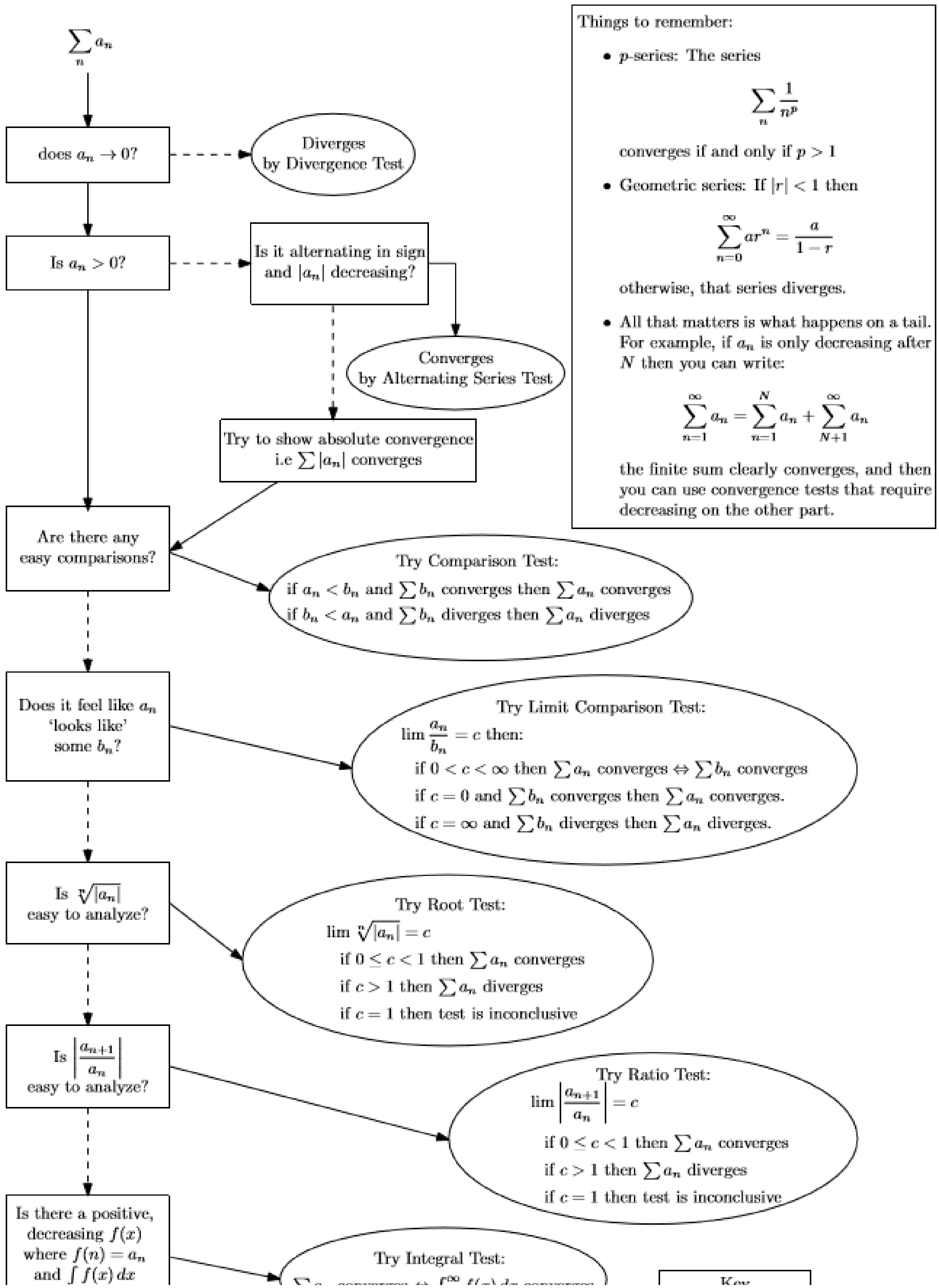
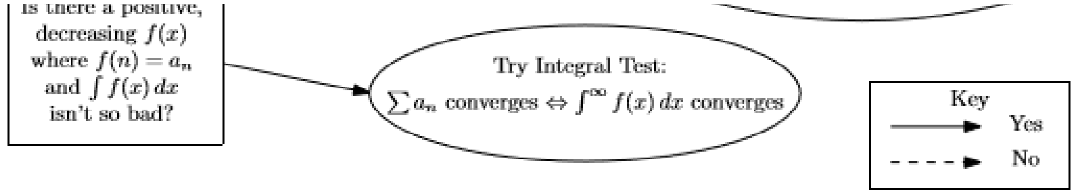
Series Convergence Flowchart




Is there a positive,
decreasing $f(x)$
where $f(n) = a_n$
and $\int f(x)\,dx$
isn't so bad?
Try Integral Test:
$\sum a_n$ converges $\Leftrightarrow \int^{\infty} f(x)\,dx$ converges
Key
Yes
No

Comparison Test. $0 < a_n < b_n$, and $\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n$ (1) (2) | (a) if (2) converges, then (1) converges as well! (b) if (1) diverges, then (2) is divergent.

1. Determine whether a series is convergent or divergent. State the test used to conclude.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$

Comparison Test.

$n^3+8 > n^3$

$\frac{1}{n^3+8} < \frac{1}{n^3} \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^3}$ - convergent p-series $p = 3 > 1$

By the comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ is convergent.

(b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

Going to compare with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$, p series $p = \frac{1}{2} < 1$, divergent.

$\sqrt{n}-1 < \sqrt{n}$, then $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$, and $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} < \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, then $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ is divergent by the Comparison Test.

(c) $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

$3+10^n > 10^n$

$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n} < \sum_{n=1}^{\infty} \frac{9^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n \leftarrow$ geometric, $r = \frac{9}{10} < 1$, so it is convergent.

By the comparison Test, $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ is convergent.

(d) $\sum_{n=1}^{\infty} \frac{1+\cos^2 n}{e^n}$

$0 \le \cos^2 n \le 1$

$\frac{1}{e^n} \le \frac{1+\cos^2 n}{e^n} \le \frac{2}{e^n} \longrightarrow \sum_{n=1}^{\infty} \frac{2}{e^n} = \sum_{n=1}^{\infty} 2\cdot\left(\frac{1}{e}\right)^n$ geometric series, $r = \frac{1}{e} < 1$

$\sum_{n=1}^{\infty} 2\cdot\frac{1}{e^n}$ is convergent.

By the Comparison Test, $\sum_{n=1}^{\infty} \frac{1+\cos^2 n}{e^n}$ is convergent.

Limit Comp. Test.

$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$

$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$

$\frac{a_n}{b_n} \neq \infty$

Then these series either both converge or both diverge

(e) $\sum_{n=1}^{\infty} \underbrace{\frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}}_{=a_n}$

$\frac{\sqrt{n}}{\sqrt{n^3+4n+3}} = \frac{n^{1/3}}{n^{3/2}} = \frac{1}{n^{3/2-1/3}} = \underbrace{\frac{1}{n^{7/6}}}_{=b_n}$, $7/6 > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$ is convergent.

$$\lim_{n\to\infty} \frac{\frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}}{\frac{1}{n^{7/6}}} = \lim_{n\to\infty} \frac{n^{7/6}\cdot\sqrt[3]{n}}{\sqrt{n^3+4n+3}} = \lim_{n\to\infty} \frac{a_n}{b_n} = 1 < \infty$$

$1 \neq 0$.

Series $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}$ is convergent by the Limit Comparison Test.

(f) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}}_{=a_n}$

$b_n = \dfrac{n \cdot n^2}{n \cdot n^4} = \dfrac{1}{n^2}$

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ p-series, } p=2 \text{ convergent.}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2} \cdot \frac{n^2}{1} = 2 > 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$ is convergent by the limit Comparison Test.

(g) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^2 e^{-n}}_{=a_n}$

$b_n = e^{-n}$, $\displaystyle\sum_{n=1}^{\infty} e^{-n}$, geometric $r = \frac{1}{e} < 1$

$\displaystyle\sum_{n=1}^{\infty} e^{-n}$ is convergent.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2 \cancel{e^{-n}}}{\cancel{e^{-n}}} = \lim_{n\to\infty} (1+\tfrac{1}{n})^2 = 1$$

$\displaystyle\sum_{n=1}^{\infty} (1+\tfrac{1}{n})^2 e^{-n}$ is convergent by the limit comparison Test.

$\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3}$ $\quad a_n = \dfrac{\sqrt{n}}{n+3}$, $\quad b_n = \dfrac{1}{\sqrt{n}}$ $\quad b_n = \dfrac{1}{n}$ bad choice!

What if I've chosen

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+3} \cdot n = \lim_{n\to\infty} \frac{n^{3/2}}{n+3} = \infty$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+3} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{n}{n+3} = 1$$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ — divergent by the Divergence Test.

Ratio Test.

$\displaystyle\sum_{n=1}^{\infty} a_n$

$\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

- $L<1$, the series converges absolutely
- $L>1$, the series diverges
- $L=1$, the test fails.

$a_n = \dfrac{e^n}{n^2}$, $\quad a_{n+1} = \dfrac{e^{n+1}}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)^2}}{\frac{e^n}{n^2}} = \lim_{n\to\infty} \frac{\overset{e\cdot \cancel{e^n}}{e^{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\cancel{e^n}} = \lim_{n\to\infty} \frac{e \cdot n^2}{(n+1)^2} = e > 1$$

The series diverges by the Ratio Test.

$$\lim_{n\to\infty} \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{e^n}{2n} = \lim_{n\to\infty} \frac{e^n}{2} = \infty \neq 0$$

L'Hospital's Rule twice.

(j) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ – converges absolutely by the Ratio Test.

$a_n = \frac{3^n \cdot n^2}{n!} = \frac{3^n \cdot n^{\cancel{2}}}{\cancel{n}\cdot(n-1)(n-2)\cdot\ldots\cdot 1} = \frac{n\cdot 3^n}{(n-1)!}$

$a_{n+1} = \frac{(n+1)\cdot 3^{n+1}}{n!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)\cdot 3^{n+1}}{n!}}{\frac{n\cdot 3^n}{(n-1)!}} = \lim_{n\to\infty} \frac{(n+1)\cdot 3^{n+1}}{n!} \cdot \frac{(n-1)!}{n\cdot 3^n} = \lim_{n\to\infty} \frac{(n+1)\overbrace{3^{n+1}}^{3\cdot\cancel{3^n}}}{n\cdot n\cancel{3^n}} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0 < 1$

$\frac{(n-1)!}{n!} = \frac{\cancel{(n-1)(n-2)\cdots 1}}{n\cancel{(n-1)(n-2)\cdots 1}} = \frac{1}{n}$

Root Test.

$\sum_{n=1}^{\infty} a_n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

- $L > 1$, the series diverges
- $L < 1$, the series converges absolutely
- $L = 1$, the test fails.

(k) $\sum_{n=1}^{\infty} \frac{2^{n-1}3^{n+1}}{n^n}$ converges absolutely by the Root Test.

$a_n = \frac{2^{n-1}\cdot 3^{n+1}}{n^n} = \frac{2^n \cdot 2^{-1}\cdot 3^n \cdot 3}{n^n} = \frac{3}{2}\cdot\frac{2^n\cdot 3^n}{n^n} = \frac{3}{2}\cdot\left(\frac{2\cdot 3}{n}\right)^n = \frac{3}{2}\left(\frac{6}{n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3}{2}\left(\frac{6}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^{1/n}\cdot\frac{6}{n} = 0 < 1$

The Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1} a_n$ is divergent.

(l) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n^2)^n}$ – diverges by the Divergence Test.

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{(1+n^2)^n}} = \lim_{n\to\infty} \frac{n^2}{(1+n^2)} = 1$

The Root Test fails.

$\lim_{n\to\infty} \frac{n^{2n}}{(1+n^2)^n} = \lim_{n\to\infty} \left(\overbrace{\cancel{\frac{n^2}{1+n^2}}}^{1}\right)^n = \lim_{n\to\infty} 1^n = 1 \neq 0$

if $\sum_{n=1}^{\infty}|a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Alternating Series Test.

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$

- $b_{n+1}<b_n$
- $\lim_{n\to\infty} b_n=0$

Then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent

2. Which of the following series is absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+3}$ an alternating series., $b_n=\left|\frac{(-1)^n}{5n+3}\right|=\frac{1}{5n+3}$.

absolutely convergent? $\boxed{NO}$ $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{5n+3}\right|=\sum_{n=1}^{\infty}\frac{1}{5n+3}$ divergent by Comparison with $\sum_{n=1}^{\infty}\frac{1}{n}$.

Conditionally convergent? $\boxed{Yes}$ Use the Alternating Series Test.

$b_n=\frac{1}{5n+3}$, $b_{n+1}=\frac{1}{5(n+1)+3}=\frac{1}{5n+8}$

since $5n+3<5n+8$, then $\underbrace{\frac{1}{5n+3}}_{b_n}>\underbrace{\frac{1}{5n+8}}_{b_{n+1}}$ $\rightarrow b_{n+1}<b_n$

$\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{5n+3}=0$

The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+3}$ is conditionally convergent by the Alternating Series Test.

(b) $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$

$\lim_{n\to\infty}\left|(-1)^n\frac{3n-1}{2n+1}\right|=\lim_{n\to\infty}\frac{3n-1}{2n+1}=\frac{3}{2}\neq 0$

Divergent by the Divergence Test.

(c) $\sum_{n=1}^{\infty}\frac{(-3)^n(n+1)}{2^{2n+1}}$, Ratio Test for $a_n=\frac{(-3)^n(n+1)}{2^{2n+1}}$, $a_{n+1}=\frac{(-3)^{n+1}(n+2)}{2^{2(n+1)+1}}=\frac{(-3)^{n+1}(n+2)}{2^{2n+3}}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\overbrace{(-3)^{n+1}}^{-3\cdot\cancel{(-3)^n}}(n+2)}{\underbrace{2^{2n+3}}_{2^2\cdot\cancel{2^{2n+1}}}}\cdot\frac{\cancel{2^{2n+1}}}{\cancel{(-3)^n}(n+1)}\right|=\lim_{n\to\infty}\left|\frac{-3\cdot(n+2)}{4(n+1)}\right|=\frac{3}{4}\lim_{n\to\infty}\frac{n+2}{n+1}=\frac{3}{4}<1$

The series converges absolutely.

(d) $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{\ln n}$

$\lim_{n\to\infty}\left|\frac{(-1)^n\sqrt{n}}{\ln n}\right|=\lim_{n\to\infty}\frac{\sqrt{n}}{\ln n}\ \underline{\underline{\text{L'Hospital's Rule}}}\ \lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{2\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}}{2}=\infty.$

Divergent by the Divergence Test.

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=S_n+R_n$, R_n is the remainder, S_n is the n-th partial sum. | $|R_n|<b_{n+1}$

3. How many terms are required to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ so that the error is less than 0.001?

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}=S_n+R_n$

$|R_n|<10^{-3}$

$R_n<b_{n+1}=\frac{1}{(n+1)^4}$

$|R_n| < 10^{-3}$

$R_n < b_{n+1} = \frac{1}{(n+1)^4}$

$\frac{1}{(n+1)^4} < 10^{-3}$ or $\frac{1}{(n+1)^4} < \frac{1}{10^3} \rightarrow (n+1)^4 > 10^3$

$n+1 > 10^{3/4}$

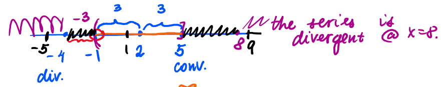
$n > 10^{3/4} - 1$

$n > 4.62\ldots$

at least $\rightarrow \boxed{n=5}$ terms.

the interval of convergence is $(-1,5]$

4. If the power series given by $\sum_{n=0}^{\infty} c_n(x-2)^n$ converges at $x = 5$ and diverges at $x = -4$, what can we say about the following series?



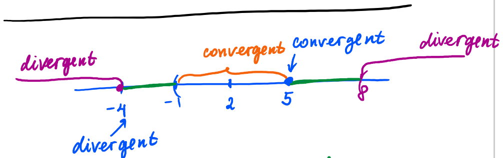
on $(-1,5]$ $\sum_{n=0}^{\infty} c_n (x-2)^n$ is convergent.

(a) $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} 1^n \cdot c_n \rightarrow$ convergent

The series $\sum_{n=0}^{\infty} c_n (x-2)^n$ is convergent for sure on $(-1,5]$

it is divergent for sure on $(-\infty,-4]\cup(8,\infty)$

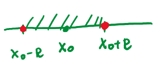
(b) $\sum_{n=0}^{\infty} c_n(-3)^n$ – not enough information.



not enough information for x on $(-4,-1)\cup(5,8)$

(c) $\sum_{n=0}^{\infty} c_n 9^n$ divergent

(d) $\sum_{n=0}^{\infty} c_n(-5)^n$ divergent.



$\sum_{n=0}^{\infty} c_n (x-x_0)^n$ | convergent inside the interval of convergence.

$|x-x_0| < R$, R is the radius of convergence.

test the endpoints $x = x_0+R$ and $x = x_0-R$ separately.

5. Find the radius of convergence and the interval of convergence.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$

Ratio Test for $a_n = (-1)^n \frac{x^n}{n^2 5^n}$, $a_{n+1} = (-1)^{n+1} \frac{x^{n+1}}{(n+1)^2 5^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{(-1)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{-x \cdot n^2}{(n+1)^2 \cdot 5} \right| = \left|\frac{x}{5}\right| \lim_{n\to\infty} \frac{\cancel{n^2}^{\,1}}{(n+1)^2} = \left|\frac{x}{5}\right| < 1$$

$\left|\frac{x}{5}\right| < 1 \rightarrow |x| < 5 \rightarrow -5 < x < 5$ interval of convergence.

The radius of convergence $\boxed{R=5}$

Test the end points.

$x=-5$: $\sum (-1)^n \frac{(-5)^n}{\ldots} = \sum \frac{5^n}{\ldots} = \sum \frac{1}{n^2}$

$x=5$: $\sum (-1)^n \frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ – convergent

$\sum_{n=1}^{\infty} a_n$

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

Test the end points.

$x=-5$

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-5)^n}{n^2 5^n}=\sum_{n=1}^{\infty}\frac{5^n}{n^2 5^n}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

convergent
p-series, $p=2>1$

$x=5$

$$\sum_{n=1}^{\infty}(-1)^n\frac{5^n}{n^2 5^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$ – convergent

by the alternating series test.

$b_n=\frac{1}{n^2}$, $b_{n+1}=\frac{1}{(n+1)^2}$

since $\frac{1}{(n+1)^2}<\frac{1}{n^2}\longrightarrow b_{n+1}<b_n$

$\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{n^2}=0$.

$\sum_{n=1}^{\infty} a_n$

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

$\boxed{\text{Interval of convergence is } [-5,5]}$

(b) $\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{(n+2)!}$

Ratio Test for $a_n=\frac{2^n(x-2)^n}{(n+2)!}$, $a_{n+1}=\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n(x-2)^n}\right|=\lim_{n\to\infty}\left|\frac{2(x-2)}{n+3}\right|=2|x-2|\lim_{n\to\infty}\frac{1}{n+3}=0<1$$

converges for all x.

$\boxed{\text{Radius of convergence } R=\infty,\ \text{interval of convergence } (-\infty,\infty)}$

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{(2x-1)^n}{5^n \sqrt{n}}$.

Ratio Test for $a_n = (-1)^n \frac{(2x-1)^n}{5^n \sqrt{n}}$, $a_{n+1} = (-1)^{n+1} \frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \frac{5^n \sqrt{n}}{(-1)^n (2x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{-1(2x-1)\sqrt{n}}{5\sqrt{n+1}} \right| = \frac{|2x-1|}{5} \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|2x-1|}{5} < 1$$

$$|2x-1| < 5$$
$$-5 < 2x-1 < 5$$
$$-4 < 2x < 6$$
$$-2 < x < 3 \quad \text{interval of convergence.}$$

End points.

$x = -2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (2(-2)-1)^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (-5)^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{divergent, } p = 1/2 < 1.$$

$x = 3$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (2\cdot 3-1)^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{- convergent by the Alternating series Test.}$$

$$b_n = \frac{1}{\sqrt{n}}, \quad \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \longrightarrow b_{n+1} < b_n$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

-2 3

$\boxed{(-2, 3]}$ interval of convergence.

$\boxed{R = \frac{|(-2,3)|}{2} = \frac{5}{2}}$ radius of convergence.