Math 150 - Week-In-Review 5 Solutions

Saud Hussein

Section 4.1 – Properties of Root Functions and Their Graphs

1. (a) **Determine the domain of $f(x) = \sqrt{6 - 5x - x^2}$.**

Solution:

Since the function involves an even root, we need $6 - 5x - x^2 \geq 0$, so

$$-(x^2 + 5x - 6) \geq 0$$
$$-(x-1)(x+6) \geq 0$$
$$(x-1)(x+6) \leq 0.$$

Creating a table and testing values around $x = -6$ and $x = 1$,

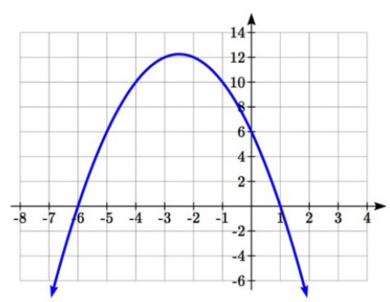
Interval	Test x	$(x-1)(x+6)$	< 0 or > 0?
$x < -6$	-7	$(-8)(-1) = 8$	greater
$-6 < x < 1$	0	$(-1)(6) = -6$	less
$x > 1$	2	$(1)(8) = 8$	greater

Therefore, the domain of f is $[-6, 1]$.

Instead of using test values and a sign diagram, we can sketch the graph of

$$g(x) = 6 - 5x - x^2.$$

Since $g(x) \geq 0$ when $-6 \leq x \leq 1$, we again find that the domain of f is $[-6, 1]$.



(b) **Sketch the graph of f, including any intercepts and asymptotes.**

Solution:

Since $f(0) = \sqrt{6}$, the vertical intercept is $(0, \sqrt{6})$.

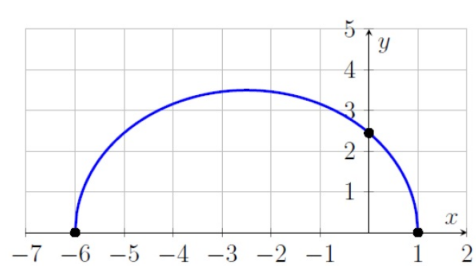
Also,

$$0 = f(x) = \sqrt{6 - 5x - x^2} = \sqrt{-(x-1)(x+6)}$$

$$0 = (x-1)(x+6),$$

so, there are horizontal intercepts at $x = 1$ and $x = -6$.

By part (a), the domain of f is $[-6, 1]$ and $f(x) \geq 0$ for an even root function.



2. (a) **Determine the domain of $f(x) = \sqrt{\dfrac{(x+2)(x-3)}{x-1}}$.**

Solution:

Since the function involves an even root of a rational expression, we need

$$\frac{(x+2)(x-3)}{x-1} \geq 0, \qquad x \neq 1.$$

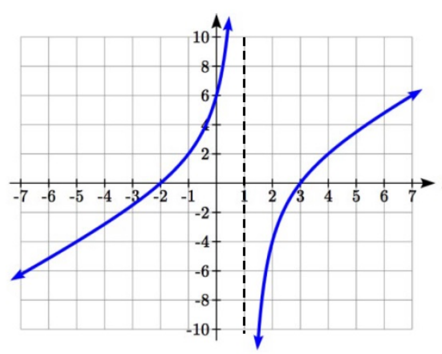
Creating a table and testing values around $x = -2$, $x = 1$, and $x = 3$,

Interval	Test x	$(x+2)(x-3)/(x-1)$	< 0 or > 0?
$x < -2$	-3	$(-1)(-6)/(-4) = -3/2$	less
$-2 < x < 1$	0	$(2)(-3)/(-1) = 6$	greater
$1 < x < 3$	2	$(4)(-1)/1 = -4$	less
$x > 3$	4	$(6)(1)/3 = 2$	greater

Therefore, the domain of f is $[-2, 1) \cup [3, \infty)$.

Instead of using test values and a sign diagram, we can sketch the graph of

$$g(x) = \frac{(x+2)(x-3)}{x-1}.$$



Since $g(x) \geq 0$ when $-2 \leq x < 1$ or $x \geq 3$, we again find that the domain of f is $[-2, 1) \cup [3, \infty)$.

(b) **Sketch the graph of f, including any intercepts and asymptotes.**

Solution:

Since $f(0) = \sqrt{6}$, the vertical intercept is $(0, \sqrt{6})$.

Also,

$$0 = f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$$

$$0 = (x+2)(x-3),$$

so, there are horizontal intercepts at $x = -2$ and $x = 3$.

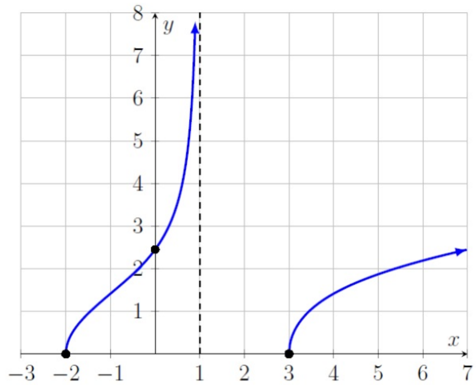
By part (a), the domain of f is $[-2, 1) \cup [3, \infty)$, and as $x \to 1^-$, $f(x) \to \infty$.

Therefore, there is a vertical asymptote at $x = 1$.

We also know that for an even root function, $f(x) \geq 0$.

Since the degree of the polynomial in the numerator of the rational expression is larger than the degree in the denominator, as

$$x \to \infty, \qquad f(x) \to \infty.$$



3. (a) **Determine the domain of $f(x) = \dfrac{2x}{\sqrt[3]{x^3 - 27}}$.**

Solution:

Since the function involves an odd root of a rational expression,

$$x^3 - 27 \neq 0$$

$$x^3 \neq 27$$

$$x \neq 3.$$

Therefore, the domain of f is $(-\infty, 3) \cup (3, \infty)$.

(b) **Sketch the graph of f, including any intercepts and asymptotes.**

Solution:

Since $f(0) = 0$, the vertical intercept is $(0, 0)$.

Also,

$$0 = f(x) = \frac{2x}{\sqrt[3]{x^3 - 27}}$$

$$0 = 2x,$$

so, the only horizontal intercept is at $x = 0$.

Creating a table and testing values around $x = 0$ and $x = 3$,

Interval	Test x	$f(x)$	< 0 or > 0?
$x < 0$	-1	$(-2)/\sqrt[3]{-28} = 2/\sqrt[3]{28}$	greater
$0 < x < 3$	1	$2/\sqrt[3]{-26} = -2/\sqrt[3]{26}$	less
$x > 3$	4	$8/\sqrt[3]{37}$	greater

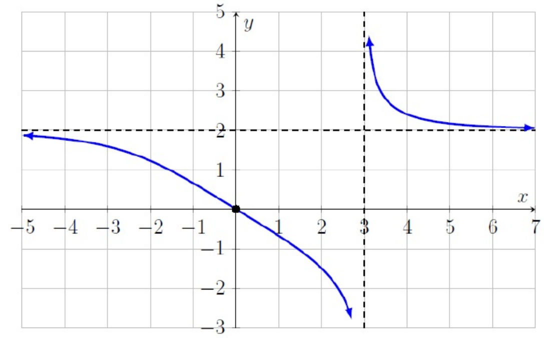
Using the table, as $x \to 3^-$, $f(x) \to -\infty$, and as $x \to 3^+$, $f(x) \to \infty$.

Therefore, there is a vertical asymptote at $x = 3$.

By part (a), the domain of f is $(-\infty, 3) \cup (3, \infty)$, and as $x \to \pm\infty$,

$$f(x) = \frac{2x}{\sqrt[3]{x^3 - 27}} \approx \frac{2x}{\sqrt[3]{x^3}} = \frac{2x}{x} \to 2.$$

Therefore, there is a horizontal asymptote at $y = 2$.



4. (a) **Determine the domain of $f(x) = \dfrac{2x}{\sqrt{x^2 - 16}}$.**

Solution:

Since the function involves an even root, we need $x^2 - 16 > 0$, so

$$(x - 4)(x + 4) > 0.$$

Creating a table and testing values around $x = -4$ and $x = 4$,

Interval	Test x	$(x-4)(x+4)$	< 0 or > 0?
$x < -4$	-5	$(-9)(-1) = 9$	greater
$-4 < x < 4$	0	$(-4)(4) = -16$	less
$x > 4$	5	$(1)(9) = 9$	greater

Therefore, the domain of f is $(-\infty, -4) \cup (4, \infty)$.

(b) **Sketch the graph of f, including any intercepts and asymptotes.**

Solution:

Since the domain does not include $x = 0$, the function does not cross the x or y-axis.

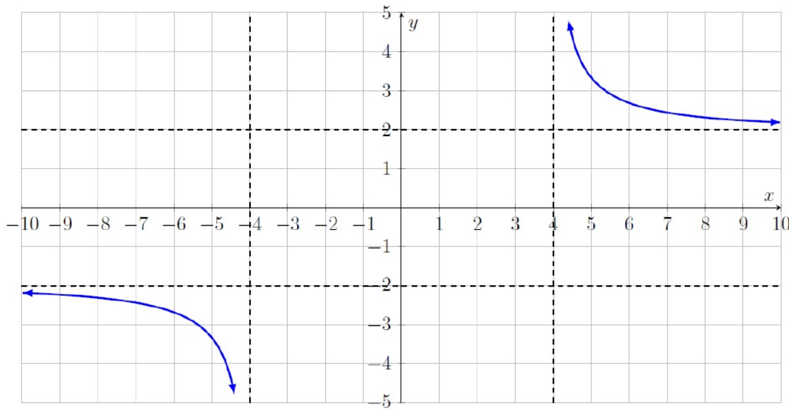
Using the table, as $x \to -4^-$, $f(x) \to -\infty$, and as $x \to 4^+$, $f(x) \to \infty$.

Therefore, there are vertical asymptotes at $x = -4$ and $x = 4$.

As $x \to \pm\infty$,

$$f(x) = \frac{2x}{\sqrt{x^2 - 16}} \approx \frac{2x}{\sqrt{x^2}} = \frac{2x}{|x|} \to \pm 2.$$

Therefore, there are also horizontal asymptotes at $y = -2$ and $y = 2$.



5. **Park rangers may construct rock piles to mark trails or other landmarks. A mound of gravel in the shape of a right circular cone with the height equal to twice the radius of the base is constructed. The volume V of such a cone as a function of the radius r is given by**

$$V(r) = \frac{2}{3}\pi r^3.$$

Determine the radius of the mound of gravel if the volume is 100 ft^3.

Solution:

Let $V = V(r)$. Solving for the radius r,

$$V = V(r) = \frac{2}{3}\pi r^3 \Rightarrow \frac{3V}{2\pi} = r^3$$

$$r = r(V) = \sqrt[3]{\frac{3V}{2\pi}}.$$

Evaluating the function at $V = 100$,

$$r(100) = \sqrt[3]{\frac{3(100)}{2\pi}} = \sqrt[3]{\frac{150}{\pi}} \approx 3.63 \text{ feet.}$$

Section 4.3 – Solving Equations Involving Root and Power Functions

1. **Solve the following radical equations.**

(a) $\sqrt{15-2x} = x$

Solution:

$$\sqrt{15-2x} = x$$
$$\left(\sqrt{15-2x}\right)^2 = x^2$$
$$15 - 2x = x^2$$
$$x^2 + 2x - 15 = 0$$
$$(x-3)(x+5) = 0$$

Therefore, $x = 3$ and $x = -5$ are potential solutions to the radical equation.

Checking $x = -5$,

$$\sqrt{15-2x} = x$$
$$\sqrt{15-2(-5)} \stackrel{?}{=} -5$$
$$\sqrt{15+10} \stackrel{?}{=} -5$$
$$\sqrt{25} \stackrel{?}{=} -5$$
$$5 \neq -5.$$

This means $x = -5$ is an extraneous solution and does not solve the equation.

Checking $x = 3$,

$$\sqrt{15-2x} = x$$
$$\sqrt{15-2(3)} \stackrel{?}{=} 3$$
$$\sqrt{15-6} \stackrel{?}{=} 3$$
$$\sqrt{9} \stackrel{?}{=} 3$$
$$3 = 3.$$

Therefore, $x = 3$ is the solution to the radical equation.

(b) $\sqrt[3]{2x-4}+7=5$

Solution:

$$\sqrt[3]{2x-4}+7=5$$
$$\sqrt[3]{2x-4}=-2$$
$$\left(\sqrt[3]{2x-4}\right)^3=(-2)^3$$
$$2x-4=-8$$
$$2x=-4$$
$$x=-2$$

Since we raised both sides to an odd power, there is no need to check the solution.

(c) $\sqrt{3x+7}+\sqrt{x+2}=1$

Solution:

$$\sqrt{3x+7}+\sqrt{x+2}=1$$
$$\sqrt{3x+7}=1-\sqrt{x+2}$$
$$\left(\sqrt{3x+7}\right)^2=\left(1-\sqrt{x+2}\right)^2$$
$$3x+7=1+(x+2)-2\sqrt{x+2}$$
$$3x+7=x+3-2\sqrt{x+2}$$
$$2x+4=-2\sqrt{x+2}$$
$$x+2=-\sqrt{x+2}$$

$$x+2=-\sqrt{x+2}$$
$$(x+2)^2=\left(-\sqrt{x+2}\right)^2$$
$$x^2+4x+4=x+2$$
$$x^2+3x+2=0$$
$$(x+1)(x+2)=0$$

Therefore, $x=-1$ and $x=-2$ are potential solutions to the radical equation.

Checking $x=-1$,

$$\sqrt{3x+7}+\sqrt{x+2}=1$$
$$\sqrt{3(-1)+7}+\sqrt{-1+2}\stackrel{?}{=}1$$
$$\sqrt{4}+\sqrt{1}\stackrel{?}{=}1$$
$$2+1\stackrel{?}{=}1$$
$$3\neq 1.$$

This means $x=-1$ is an extraneous solution and does not solve the equation.

Checking $x = -2$,

$$\sqrt{3x+7} + \sqrt{x+2} = 1$$
$$\sqrt{3(-2)+7} + \sqrt{-2+2} \stackrel{?}{=} 1$$
$$\sqrt{1} + \sqrt{0} \stackrel{?}{=} 1$$
$$1 + 0 \stackrel{?}{=} 1$$
$$1 = 1.$$

Therefore, $x = -2$ is the solution to the radical equation.

(d) $\sqrt{2x+3} - \sqrt{x+1} = 1$

Solution:

$$\sqrt{2x+3} - \sqrt{x+1} = 1$$
$$\sqrt{2x+3} = 1 + \sqrt{x+1}$$
$$\left(\sqrt{2x+3}\right)^2 = \left(1 + \sqrt{x+1}\right)^2$$
$$2x + 3 = 1 + (x+1) + 2\sqrt{x+1}$$
$$2x + 3 = x + 2 + 2\sqrt{x+1}$$
$$x + 1 = 2\sqrt{x+1}$$
$$(x+1)^2 = \left(2\sqrt{x+1}\right)^2$$
$$x^2 + 2x + 1 = 4x + 4$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$

Therefore, $x = -1$ and $x = 3$ are potential solutions to the radical equation.

Checking $x = -1$,

$$\sqrt{2x+3} - \sqrt{x+1} = 1$$
$$\sqrt{2(-1)+3} - \sqrt{-1+1} \stackrel{?}{=} 1$$
$$\sqrt{1} - \sqrt{0} \stackrel{?}{=} 1$$
$$1 - 0 \stackrel{?}{=} 1$$
$$1 = 1.$$

Therefore, $x = -1$ is a solution to the radical equation.

Checking $x = 3$,

$$\sqrt{2x+3} - \sqrt{x+1} = 1$$
$$\sqrt{2(3)+3} - \sqrt{3+1} \stackrel{?}{=} 1$$
$$\sqrt{9} - \sqrt{4} \stackrel{?}{=} 1$$
$$3 - 2 \stackrel{?}{=} 1$$
$$1 = 1.$$

Therefore, $x = 3$ is also a solution to the radical equation.

(e) $x^{5/4} = 32$

Solution:

$$x^{5/4} = 32$$
$$(\sqrt[4]{x})^5 = 32$$
$$\sqrt[4]{x} = \sqrt[5]{32}$$
$$\sqrt[4]{x} = 2$$
$$(\sqrt[4]{x})^4 = 2^4$$
$$x = 16$$

Therefore, $x = 16$ is a potential solution to the radical equation.

Checking $x = 16$,

$$x^{5/4} = 32$$
$$(\sqrt[4]{x})^5 = 32$$
$$\left(\sqrt[4]{16}\right)^5 \stackrel{?}{=} 32$$
$$(2)^5 \stackrel{?}{=} 32$$
$$32 = 32.$$

Therefore, $x = 16$ is the solution to the radical equation.

(f) $(2x+3)^{2/3} = 9$

Solution:

$$(2x+3)^{2/3} = 9$$
$$\left(\sqrt[3]{2x+3}\right)^2 = 9$$
$$\sqrt[3]{2x+3} = \pm\sqrt{9}$$
$$\sqrt[3]{2x+3} = \pm 3$$
$$\left(\sqrt[3]{2x+3}\right)^3 = (\pm 3)^3$$
$$2x+3 = \pm 27$$

$$2x+3 = \pm 27$$
$$2x = -3 \pm 27$$
$$x = \frac{-3 \pm 27}{2}$$

Therefore, $x = -\frac{30}{2} = -15$ and $x = \frac{24}{2} = 12$ are solutions to the radical equation.

Since we raised both sides to an odd power, the solutions are not extraneous.

(g) $(x-1)^{3/4} = -27$

Solution:

$$(x-1)^{3/4} = -27$$
$$\left(\sqrt[4]{x-1}\right)^3 = -27$$
$$\sqrt[4]{x-1} = \sqrt[3]{-27}$$
$$\sqrt[4]{x-1} = -3$$
$$\left(\sqrt[4]{x-1}\right)^4 = (-3)^4$$
$$x-1 = 81$$

Therefore, $x = 82$ is a potential solution to the radical equation.

Checking $x = 82$,

$$\left(\sqrt[4]{x-1}\right)^3 = -27$$
$$\left(\sqrt[4]{82-1}\right)^3 \stackrel{?}{=} -27$$
$$\left(\sqrt[4]{81}\right)^3 \stackrel{?}{=} -27$$
$$(3)^3 \stackrel{?}{=} -27$$
$$27 \neq -27.$$

Therefore, $x = 82$ is an extraneous solution, there are no solutions to the equation.

Section 4.4 – Solving Nonlinear Inequalities

1. **Solve the following inequalities.**

(a) $x^2 > x + 12$

Solution:

$$x^2 > x + 12$$
$$x^2 - x - 12 > 0$$
$$(x-4)(x+3) > 0$$

Creating a table and testing values around $x = -3$ and $x = 4$,

Interval	Test x	$(x-4)(x+3)$	< 0 or > 0?
$x < -3$	-4	$(-8)(-1) = 8$	greater
$-3 < x < 4$	0	$(-4)(3) = -12$	less
$x > 4$	5	$(1)(8) = 8$	greater

Therefore, the inequality is true over the interval $(-\infty, -3) \cup (4, \infty)$.

(b) $4x^2 + 1 \leq 4x$

Solution:

$$4x^2 + 1 \leq 4x$$
$$4x^2 - 4x + 1 \leq 0$$
$$(2x-1)^2 \leq 0$$

Since squaring a number never produces a negative value and

$$2x - 1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2},$$

the inequality is true only when $x = 1/2$.

(c) $2x - x^2 \geq |x-1| - 1$

Solution:

Using the piecewise definition of the absolute value,

$$2x - x^2 \geq |x-1| - 1$$

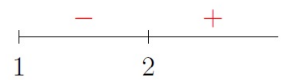
$$2x - x^2 \geq \begin{cases} (x-1) - 1, & x - 1 \geq 0 \\ -(x-1) - 1, & x - 1 < 0 \end{cases}$$

$$2x - x^2 \geq \begin{cases} x - 2, & x \geq 1 \\ -x, & x < 1 \end{cases}$$

$$0 \geq \begin{cases} x^2 - x - 2, & x \geq 1 \\ x^2 - 3x, & x < 1 \end{cases}.$$

For $x \geq 1$,

$$0 \geq x^2 - x - 2$$
$$0 \geq (x-2)(x+1).$$



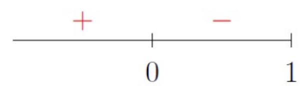
Since $x \geq 1$, creating a table and testing values around $x = 2$,

Interval	Test x	$(x-2)(x+1)$	< 0 or > 0?
$1 < x < 2$	1.5	$(-0.5)(2.5) = -1.25$	less
$x > 2$	3	$(1)(4) = 4$	greater

Therefore, this inequality is true over the interval $[1, 2]$.

For $x < 1$,

$$0 \geq x^2 - 3x$$
$$0 \geq x(x-3).$$



Since $x < 1$, creating a table and testing values around $x = 0$,

Interval	Test x	$x(x-3)$	< 0 or > 0?
$x < 0$	-1	$(-1)(-4) = 4$	greater
$0 < x < 1$	0.5	$(0.5)(-2.5) = -1.25$	less

This inequality is true over the interval $[0, 1)$, therefore, the absolute value inequality is true over the interval $[0, 1) \cup [1, 2] = [0, 2]$.

(d) $-\dfrac{6x+6}{x^2-x-2} \leq x+3$

Solution:

If $x \neq -1$,

$$-\frac{6x+6}{x^2-x-2} \leq x+3$$

$$-\frac{6\cancel{(x+1)}}{(x-2)\cancel{(x+1)}} \leq x+3$$

$$-\frac{6}{x-2} \leq x+3$$

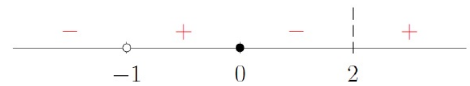
$$-\frac{6}{x-2} \leq \frac{(x+3)(x-2)}{x-2}$$

$$-\frac{6}{x-2} \leq \frac{(x+3)(x-2)}{x-2}$$

$$0 \leq \frac{x^2+x-6}{x-2} + \frac{6}{x-2}$$

$$0 \leq \frac{x^2+x}{x-2}$$

$$0 \leq \frac{x(x+1)}{x-2}$$



To determine the solution to the inequality $0 \leq \frac{x(x+1)}{x-2}$, we must then test values around $x = -1$, $x = 0$, and $x = 2$,

Interval	Test x	$x(x+1)/(x-2)$	< 0 or > 0?
$x < -1$	-2	$(-2)(-1)/(-4) = -0.5$	less
$-1 < x < 0$	-0.5	$(-0.5)(0.5)/(-2.5) = 0.1$	greater
$0 < x < 2$	1	$(1)(2)/(-1) = -2$	less
$x > 2$	3	$(3)(4)/(1) = 12$	greater

Therefore, the inequality is true over the interval $(-1, 0] \cup (2, \infty)$.

(e) $\sqrt[3]{x} \leq x$

Solution:

Since the cubic function is increasing,

$$\sqrt[3]{x} \leq x$$

$$(\sqrt[3]{x})^3 \leq x^3$$

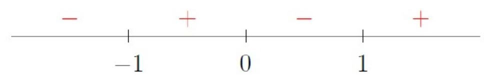
$$x \leq x^3$$

$$0 \leq x^3 - x$$

$$0 \leq x(x^2-1)$$

$$0 \leq x(x-1)(x+1)$$

To determine the solution to the inequality $0 \leq x(x-1)(x+1)$, we then test values around $x=-1$, $x=0$, and $x=1$,



Interval	Test x	$x(x-1)(x+1)$	<0 or >0?
$x<-1$	-2	$(-2)(-3)(-1)=-6$	less
$-1<x<0$	-0.5	$(-0.5)(-1.5)(0.5)=0.375$	greater
$0<x<1$	0.5	$(0.5)(-0.5)(1.5)=-0.375$	less
$x>1$	2	$(2)(1)(3)=6$	greater

Therefore, the inequality is true over the interval $[-1,0]\cup[1,\infty)$.

(f) $2(x-2)^{-1/3}-\frac{2}{3}x(x-2)^{-4/3}\leq 0$

Solution:

$$2(x-2)^{-1/3}-\frac{2}{3}x(x-2)^{-4/3}\leq 0$$

$$\frac{2}{(x-2)^{1/3}}-\frac{2x}{3(x-2)^{4/3}}\leq 0$$

$$\frac{3(x-2)}{3(x-2)}\cdot\frac{2}{(x-2)^{1/3}}-\frac{2x}{3(x-2)^{4/3}}\leq 0$$

$$\frac{6x-12}{3(x-2)^{4/3}}-\frac{2x}{3(x-2)^{4/3}}\leq 0$$

$$\frac{6x-12}{3(x-2)^{4/3}}-\frac{2x}{3(x-2)^{4/3}}\leq 0$$

$$\frac{4(x-3)}{3\left(\sqrt[3]{x-2}\right)^4}\leq 0$$

If $x\neq 2$, then

$$x-3\leq 0.$$

Therefore, the inequality is true when $x\leq 3$ and $x\neq 2$, so over the interval

$$(-\infty,2)\cup(2,3].$$

Instead, factoring out $(x-2)^{-4/3}$,

$$2(x-2)^{-1/3} - \frac{2}{3}x(x-2)^{-4/3} \leq 0$$

$$6(x-2)^{-1/3} - 2x(x-2)^{-4/3} \leq 0$$

$$2(x-2)^{-4/3}[3(x-2)^1 - x] \leq 0$$

$$2(x-2)^{-4/3}[2x-6] \leq 0$$

$$\frac{4(x-3)}{(x-2)^{4/3}} \leq 0.$$

