STAT 201 - Week-In-Review 5

Dr. Prasenjit Ghosh

Problem Solutions

1. Consider the random experiment of flipping a fair coin thrice.

   (a) Write down the sample space $\mathcal{S}$ in this context.

   Solution:
   $$\mathcal{S} = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}.$$

   (b) Define the event A as obtaining exactly one head. Enumerate the elements of A.

   Solution:
   $$A = \{\text{HTT, THT, TTH}\}.$$

   (c) Define the event B as obtaining a head in the second flip. Enumerate the elements of B.

   Solution:
   $$B = \{\text{HHH, HHT, THH, THT}\}.$$

   (d) Are the events A and B disjoint?

   Solution: No, because $\text{A} \cap \text{B} = \{\text{THT}\} \neq \emptyset$.

   (e) Find the probability that exactly one head appeared and the second flip resulted in a head.

   Solution: Need to find $\mathrm{P}(\text{A} \cap \text{B})$.

   Observe that $\text{A} \cap \text{B} = \{\text{THT}\}$. Since the coin is fair, the elements of the sample space $\mathcal{S}$ are equally likely. Hence, using the classical definition, we obtain, $\mathrm{P}(\text{A} \cap \text{B}) = 1/8$.

   (f) Find the probability that either exactly one head appeared or the second flip resulted in a head.

   Solution: Need to find $\mathrm{P}(\text{A} \cup \text{B})$.

   Observe that $\text{A} \cup \text{B} = \{\text{HTT, THT, TTH, THH, HHT, HHH}\}$. Since the elements of the sample space $\mathcal{S}$ are equally likely, using the classical definition, we obtain, $\mathrm{P}(\text{A} \cup \text{B}) = \frac{6}{8}$.

   Alternatively, using the union law, we obtain
   $$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{3}{8} + \frac{4}{8} - \frac{1}{8} = \frac{6}{8}.$$

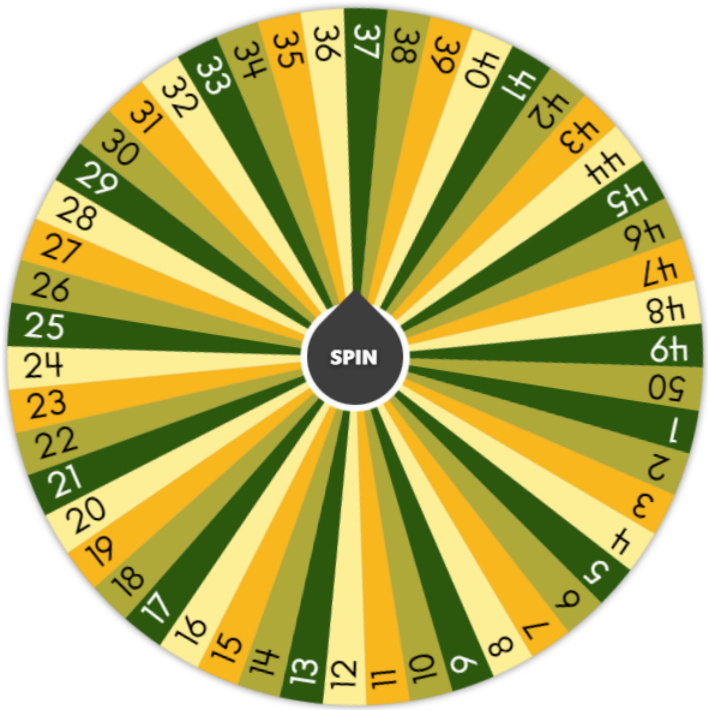
   (g) Find the probability that exactly one head appeared and the second flip resulted in a tail.

   Solution: Need to find $\mathrm{P}(\text{A} \cap \text{B}^c) = \mathrm{P}(\text{A} - \text{B})$.

   Since $\text{A} - \text{B} = \{\text{HTT, TTH}\}$, and the elements of the sample space $\mathcal{S}$ are equally likely, using the classical definition, we obtain, $\mathrm{P}(\text{A} - \text{B}) = 2/8$.

   Alternatively, using the probability law $\mathrm{P}(\text{A} - \text{B}) = \mathrm{P}(\text{A}) - \mathrm{P}(\text{A} \cap \text{B})$ we obtain
   $$\mathrm{P}(A - B) = \mathrm{P}(A) - \mathrm{P}(A \cap B) = \frac{3}{8} - \frac{1}{8} = \frac{2}{8}.$$

2. Consider the random experiment of drawing a number at random from the set of first 50 positive whole numbers using the random number wheel.



Define, two events A and B as follows:

A = "the number drawn is an even number (that is, a multiple of 2)"
B = "the number drawn is a multiple of 3"

(a) Are the events A and B disjoint?

Solution: Observe that A = $\{2, 4, 6, 8, \ldots, 48, 50\}$ and B = $\{3, 6, 9, 12, \ldots, 45, 48\}$. So, $A \cap B = \{6, 12, 18, 24, \ldots, 42, 48\} \neq \emptyset$. Hence, events A and B are not disjoint.

(b) Find the probability that the number drawn is a multiple of 6.

Solution: Need to find $\mathrm{P}(A \cap B)$. Here, the sample space $\mathcal{S} = \{1, 2, 3, \ldots, 49, 50\}$ consists of 50 equally likely elementary outcomes. Of these 50 equally likely elements of the sample space, 8 favor the occurrence of event $A \cap B = \{6, 12, 18, \ldots, 48\}$. Therefore, using the classical definition, we obtain $\mathrm{P}(A \cap B) = 8/50 = 4/25$.

(c) Identify the event that the number drawn is an odd number (that is, not a multiple of 2). Find its probability.

Solution: Here, $A^c = \{1, 3, 5, 7, \ldots, 47, 49\}$ denotes the event that the number drawn is an odd number.

By the classical definition, $\mathrm{P}(A) = 25/50 = 0.5$. Now using the fact $\mathrm{P}(A^c) = 1 - \mathrm{P}(A)$, we obtain $\mathrm{P}(A^c) = 1 - 0.5 = 0.5$.

3. In a certain community, 60% of the families own a dog, 70% own a cat, and 50% own both a dog and a cat.

   (a) What is the probability that a randomly selected family owns either a dog or a cat?

   Solution: Let D and C denote the events that a randomly selected family own a dog, and that a randomly selected family own a cat, respectively.

   According to the problem,

   $$\mathrm{P}(D) = 0.60,\ \mathrm{P}(C) = 0.70,\ \text{and}\ \mathrm{P}(D \cap C) = 0.50$$

   Using the Union law, we obtain:

   $$\mathrm{P}(D \cup C) = \mathrm{P}(D) + \mathrm{P}(C) - \mathrm{P}(D \cap C) = 0.60 + 0.70 - 0.50 = 0.80.$$

   (b) What is the probability that a randomly selected family owns a dog but not a cat?

   Solution: Here, D − C denotes the event that a randomly selected family owns a dog but not a cat. Hence,

   $$\mathrm{P}(D - C) = \mathrm{P}(D) - \mathrm{P}(D \cap C) = 0.60 - 0.50 = 0.10$$

   (c) What is the probability that a randomly selected family owns a cat but not a dog.

   Solution: Here, C − D denotes the event that a randomly selected family owns a cat but not a dog. Hence,

   $$\mathrm{P}(C - D) = \mathrm{P}(C) - \mathrm{P}(D \cap C) = 0.70 - 0.50 = 0.20$$

   (d) What is the probability that a randomly selected family owns exactly one of these two kinds of pets?

   Solution: Here, (D − C) ∪ (C − D) denotes the event that a randomly selected family owns exactly one of these two kinds of pets. Again, the events D − C and C − D are disjoint.

   Hence, using the Addition law, we obtain:

   $$\mathrm{P}\left((D - C) \cup (C - D)\right) = \mathrm{P}\left(D - C\right) + \mathrm{P}\left(C - D\right) = 0.10 + 0.20 = 0.30$$

   (e) Given that a randomly selected family owns a dog, what is the probability that the family own a cat also?

   Solution: Using the definition of conditional probability, we obtain

   $$\mathrm{P}(C|D) = \frac{\mathrm{P}(C \cap D)}{\mathrm{P}(D)} = \frac{0.50}{0.60} \approx 0.833$$

   (f) Are the events of owing a dog and owing a cat independent?

   Solution: No, because $\mathrm{P}(\mathrm{C}|\mathrm{D}) \neq \mathrm{P}(\mathrm{C})$.

4. Consider the following statements about two events A and B with $0 < \mathrm{P}(A),\ \mathrm{P}(B) < 1$:

   (I) A and B are mutually exclusive if A and B cannot occur together.
   (II) $\mathrm{P}(A - B) = \mathrm{P}(A) - \mathrm{P}(A \cap B)$
   (III) A and B are independent if $A \cap B = \emptyset$.
   (IV) A and B are mutually exclusive if they are independent.
   (V) A and B are independent if $\mathrm{P}(B|A) = \mathrm{P}(B)$.
   (VI) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

   Carefully choose the correct answer from the following options.

   (a) (I), (V), and (VI) are true.
   (b) (I), (II), and (V) are true.
   (c) (I), (II), and (VI) are true.
   (d) (I), (III), (IV), (V), and (VI) are true.
   (e) (I), (IV), (V), and (VI) are true.

5. Which of the following statements about two events A and B with $0 < \mathrm{P}(A),\ \mathrm{P}(B) < 1$ is (are) **INCORRECT**? Select all that apply.

   (a) $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)$
   (b) $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
   (c) $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A)$
   (d) $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A|B)$
   (e) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
   (f) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

6. Suppose A and B are two disjoint events with $\mathrm{P}(A) = 0.3$, and $\mathrm{P}(B) = 0.5$.

   Find the probabilities of the following events:

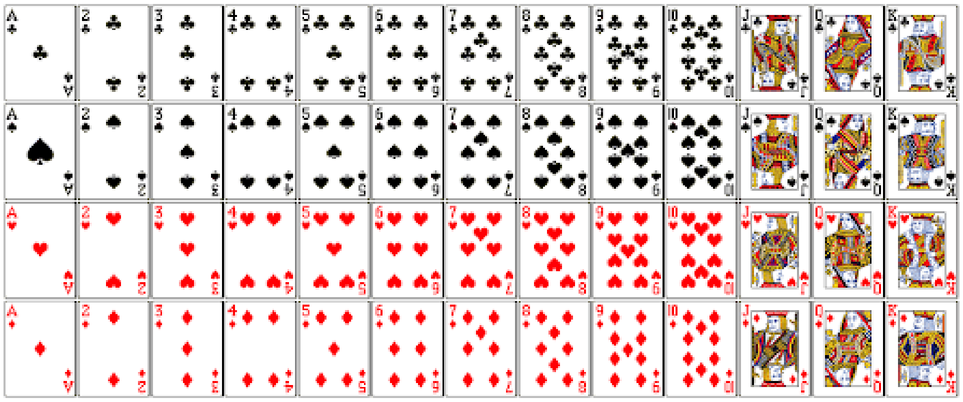
   (a) Both events A and B occur. $(A \cap B)$
   (b) At least one of events A and B occur. $(A \cup B)$
   (c) Event A occurs, but event B does not occur. $(A - B = A \cap B^c)$
   (d) Neither event A occurs, nor event B occurs. $(A^c \cap B^c)$

   Solution:

   (a) Since events A and B are disjoint, $A \cap B = \emptyset$. Hence, $\mathrm{P}(A \cap B) = 0$.
   (b) Using the Union Law, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.3 + 0.5 - 0 = 0.8$.
   (c) Since events A and B are disjoint, $A \subset B^c$. Hence, $A - B = A \cap B^c = A$ so that $\mathrm{P}(A - B) = \mathrm{P}(A) = 0.3$.
   (d) By De Moivre's law, $A^c \cap B^c = (A \cup B)^c$. Hence,

   $$\mathrm{P}\left(A^c \cap B^c\right) = \mathrm{P}\left((A \cup B)^c\right) = 1 - \mathrm{P}(A \cup B) = 1 - 0.8 = 0.2.$$

7. A card is drawn randomly from a well–shuffled deck of 52 cards.



(a) What is the probability that the card drawn is a Spade and a face card?

(b) What is the probability that the card drawn is either a Spade or a face card?

(c) What is the probability that the card drawn is neither a Spade nor a face card?

(d) Given that the the card drawn is a Spade, what is the probability that is a face card?

Solution: Observe that, here the sample space $\mathcal{S}$ is the deck of 52 cards itself, and its elements are all equally likely (each has the same common chance of $1/52$ to be drawn).

Let A and B denote the events “the card drawn is a Spade” and “the card drawn is a face card”, respectively.

(a) Observe that, of the 52 many equally likely elementary outcomes of the sample space $\mathcal{S}$, 13 favor the occurrence of event A (since there are 13 Ace cards), 12 favor the occurrence of event B (since there are 12 face cards), and 3 favor the occurrence of event $A \cap B$ (since there are 3 face cards of Ace).

Therefore, by the classical definition of probability, $\mathrm{P}(A) = 13/52$, $\mathrm{P}(B) = 12/52$, and $\mathrm{P}(A \cap B) = 3/52$.

(b) Here $A \cup B$ denotes the event that the card drawn is either an Ace or a face card. Therefore, using the Union Law, we obtain

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}.$$

(c) Here $A^c \cap B^c$ denotes the event that the card drawn is neither an Ace nor a face card.

Now, by De Moivre’s Law, we have, $A^c \cap B^c = (A \cup B)^c$.

Hence, the required probability is:

$$\mathrm{P}(A^c \cap B^c) = \mathrm{P}\left((A \cup B)^c\right) = 1 - \mathrm{P}(A \cup B) = 1 - 11/26 = 15/26.$$

(d) By the definition of conditional probability,

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \frac{(3/52)}{(13/52)} = \frac{3}{13}.$$

8. Two professors are applying for grants. Professor Jane has a probability of 0.61 of being funded. Professor Joe has probability 0.27 of being funded. Since the grants are submitted to two different federal agencies, assume the outcomes for each grant are independent.

   (a) What is the probability that both professors get their grants funded? Give your answer to four decimal places.
   (b) What is the probability that at least one of the professors will be funded? Give your answer to four decimal places.
   (c) What is the probability that Professor Jane is funded but Professor Joe is not?
   (d) Given at least one of the professors is funded, what is the probability that Professor Jane is funded but Professor Joe is not? Give your answer to four decimal places.
   (e) What is the probability that neither of the professors will be funded? Give your answer to four decimal places.

   Solution: Let A and B denote respectively the events that Prof. Jane and Prof. Joe are funded. According to the problem, $\mathrm{P}(A) = 0.61$, and $\mathrm{P}(B) = 0.27$, and that they are independent.

   (a) Here, $A \cap B$ denotes the event that both the professors get their grants funded. Now, because of the independence of A and B,

   $$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = 0.1647.$$

   (b) Here, $A \cup B$ denotes the event that at least one of the professors will be funded. Applying the Union Law, we obtain

   $$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.61 + 0.27 - 0.1647 = 0.7153.$$

   (c) Here, $A - B$ denotes the event that Professor Jane is funded but Professor Joe is not. Therefore,

   $$\mathrm{P}(A - B) = \mathrm{P}(A) - \mathrm{P}(A \cap B)) = 0.61 - 0.1647 = 0.4453.$$

   (d) By the definition of conditional probability,

   $$\mathrm{P}(A - B | A \cup B) = \mathrm{P}((A - B) \cap (A \cup B))/\mathrm{P}(A \cup B).$$

   Since the event $A - B$ is a subset of the event $A \cup B$,

   $$(A - B) \cap (A \cup B) = A - B.$$

   Using the above observation, it follows

   $$\mathrm{P}(A - B | A \cup B) = \mathrm{P}(A - B)/\mathrm{P}(A \cup B) = 0.4453/0.7153 \approx 0.6225.$$

   (e) Using De Moivre's Law, we have, $A^c \cap B^c = (A \cup B)^c$.

   Hence, the required probability is:

   $$\mathrm{P}(A^c \cap B^c) = \mathrm{P}\left((A \cup B)^c\right) = 1 - \mathrm{P}(A \cup B) = 1 - 0.7153 = 0.2847.$$

9. A large Tech Company is considering developing a generative artificial intelligence chatbot. To determine general interest in this new project, the company conducted a survey of 50,000 employees across 4 different departments. The employees were asked to rate their excitement for the project on a scale of 1-5, with 5 being the most excited. The results of the survey are provided below.

Company Department	1	2	3	4	5	TOTAL
Finance	3500	2900	1500	1600	500	**10000**
Business Administration	2000	2200	3800	2100	1900	**12000**
Research and Development	3000	300	300	300	3100	**7000**
Sales and Marketing	500	1500	3500	5000	10500	**21000**
TOTAL	**9000**	**6900**	**9100**	**9000**	**16000**	**50000**

(a) What is the probability that a randomly selected surveyed employee gave a rating of 3 or higher? Round your answer to three decimal places. ($34100/50000 = 0.682$)

(b) What is the probability that a randomly selected surveyed employee works in the 'Research and Development' Department and gave a rating of 3 or higher? Round your answer to three decimal places. ($3700/50000 = 0.074$)

(c) Given that a randomly selected surveyed employee gave a rating of 3 or higher, what is the probability that they work in the 'Research and Development' Department? Round your answer to four decimal places. ($3700/34100 = 0.1085$)

(d) Given that a randomly selected surveyed employee gave a rating of 2 or less, what is the probability that they work in the 'Research and Development' Department? Round your answer to five decimal places. ($3300/15900 = 0.20755$)

(e) Are the events of working in the 'Research and Development' Department and giving a rating of 3 or higher independent? (**No, because P(working in the 'Research and Development' Department|giving a rating of 3 or higher) = 0.1085 ≠ P(working in the 'Research and Development' Department) =** $7000/50000 = 0.14$)

10. An individual's blood type is described by the ABO system and the Rhesus factor. In the American population 16% of individuals have a negative Rhesus factor (Rh-), and 43.75% of those with Rh- have blood type O. What is the probability that a randomly selected American will have both Rh- and blood type O?

Solution: Let A and B denote respectively a randomly selected American has negative Rhesus factor (Rh-) and blood type O, respectively.

According to the problem, $\mathrm{P}(A) = 0.16, \text{ and } \mathrm{P}(B|A) = 0.4375$.

Using the Multiplication Law, we obtain

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B|A) = 0.16 \times 0.4375 = 0.07.$$

11. Fifteen percent of all births involve Cesarean (C) section. Ninety-eight percent of all babies survive delivery (S). When a C-section is performed the baby survives with probability 0.96. What is the probability that a baby will survive delivery given a C-section is not performed?

Solution: Using the Multiplication Law, we obtain

$$\mathrm{P}(S \cap C) = \mathrm{P}(S|C)\mathrm{P}(C) = 0.96 \times 0.15 = 0.144$$
$$\Rightarrow \mathrm{P}(S \cap C^c) = \mathrm{P}(S) - \mathrm{P}(S \cap C) = 0.98 - 0.144 = 0.836$$

Therefore,

$$\mathrm{P}(\mathrm{S}|\mathrm{C}^c) = \frac{\mathrm{P}(\mathrm{S} \cap \mathrm{C}^c)}{\mathrm{P}(\mathrm{C}^c)} = \frac{0.836}{0.85} = 0.9835.$$

12. Suppose we roll two fair dice together. Then the sample space would comprise of $6 \times 6 = 36$ many equally likely paired outcomes as:

$$\mathcal{S} = \left\{ \begin{matrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{matrix} \right\}$$

Define the following events:

$$\mathrm{B} = \text{the sum of two faces is an even number (i.e. a multiple of 2)}$$
$$\mathrm{C} = \text{the sum of two faces is a multiple of 3.}$$

Are the events B and C independent of each other?

Solution: Here, we shall use the **reduced or effective sample space approach**.

Note that,

$$B = \left\{ \begin{matrix} (1,1),(1,3),(1,5),(2,2),(2,4),(2,6), \\ (3,1),(3,3),(3,5),(4,2),(4,4),(4,6), \\ (5,1),(5,3),(5,5),(6,2),(6,4),(6,6) \end{matrix} \right\},$$

and

$$C = \left\{ \begin{matrix} (1,2),(1,5),(2,1),(2,4),(3,3),(3,6), \\ (4,2),(4,5),(5,1),(5,4),(6,3),(6,6) \end{matrix} \right\},$$

whence

$$B \cap C = \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,6)\}.$$

Of the 12 elements of C (which are equally likely), 6 favor the occurrence of B. Hence, by the classical definition, it readily follows

$$\mathrm{P}(B|C) = \frac{6}{12} = \frac{1}{2} = \mathrm{P}(B).$$

Alternatively, using the definition of conditional probability, you may verify that

$$\mathrm{P}(B|C) = \frac{\mathrm{P}(B \cap C)}{C} = \frac{(6/36)}{(12/36)} = \frac{6}{12} = \frac{1}{2} = \mathrm{P}(B).$$

Hence, events B and C are independent of each other.

13. The success probability of a particular knee surgery performed by a surgeon is 0.95. Suppose he operates two patients in a day. Assume that the surgeries are done independent of each other. What is the probability of exactly one successful surgery in a day?

Solution: Let S and F respectively denote the "success" and the "failure" of a given surgery. According to the problem, $\mathrm{P}(\mathrm{S}) = 0.95$ and $\mathrm{P}(\mathrm{F}) = 1 - \mathrm{P}(\mathrm{S}) = 0.05$.

Here, the sample space $\mathcal{S} = \{\mathrm{SS}, \mathrm{SF}, \mathrm{FS}, \mathrm{FF}\}$.

We need to find the probability $\mathrm{P}(\{\mathrm{SF}, \mathrm{FS}\})$.

Claim: The elements of the sample $\mathcal{S}$ are not equally likely!

Observe that because of the independence of the surgeries

$$\begin{aligned}
\mathrm{P}(\mathrm{SS}) &= 0.95 \times 0.95 = 0.9025 \\
\mathrm{P}(\mathrm{SF}) &= 0.95 \times 0.05 = 0.0475 \\
\mathrm{P}(\mathrm{FS}) &= 0.05 \times 0.95 = 0.0475 \\
\mathrm{P}(\mathrm{FF}) &= 0.05 \times 0.05 = 0.0025.
\end{aligned}$$

Hence, the elements of $\mathcal{S}$ are not equally likely.

Now, since the events $\{\mathrm{SF}\}$ and $\{\mathrm{FS}\}$ are disjoint, we have

$$\mathrm{P}(\{SF, FS\}) = \mathrm{P}(SF) + \mathrm{P}(FS) = 2 \times 0.0475 = 0.095.$$

Note: Since the elements of $\mathcal{S}$ are not equally likely, the probability of exactly one successful surgery will NOT be $\frac{1}{4}$.

14. It snows 3% of the days in Laramie. If it snows there is a 20% chance I'll get a ride to school. If it doesn't snow there is a 15% chance I'll get a ride to school. What percentage of the time that I get a ride is it snowing?

Solution: Let A be the event that it snows in Laramie on a random day. Also, let B be the event that I will get a ride to the school on a random day.

According to the problem, $\mathrm{P}(\mathrm{A}) = 0.03$, $\mathrm{P}(\mathrm{B}|\mathrm{A}) = 0.2$, and $\mathrm{P}(\mathrm{B}|\mathrm{A}^c) = 0.15$.

Need to find

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(B)} \text{ [using Bayes' Theorem]}.$$

Using the Law of Total Probability, we obtain

$$\mathrm{P}(B) = \mathrm{P}(B|A)\mathrm{P}(A) + \mathrm{P}(B|A^c)P(A^c) = 0.2 * 0.03 + 0.15 * 0.97.$$

Hence,

$$\mathrm{P}(A|B) = \frac{0.2 * 0.03}{0.2 * 0.03 + 0.15 * 0.97} \approx 0.0396.$$

15. 40% of batches of bean seeds come from Supplier A and 60% come from Supplier B. Seeds from supplier A have 50% germination rate while those from supplier B have a 75% germination rate. Given that a randomly selected batch of seeds is germinated, what is the probability that it came from supplier A?

Solution: Let A and B be the events that a randomly selected batch of seeds come from Supplier A and Supplier B, respectively. Also, let G be the event that a randomly selected batch of seeds is germinated.

According to the problem, $\mathrm{P}(A) = 0.40$, $\mathrm{P}(B) = 0.60$, $\mathrm{P}(G|A) = 0.50$, and $\mathrm{P}(G|B) = 0.75$.

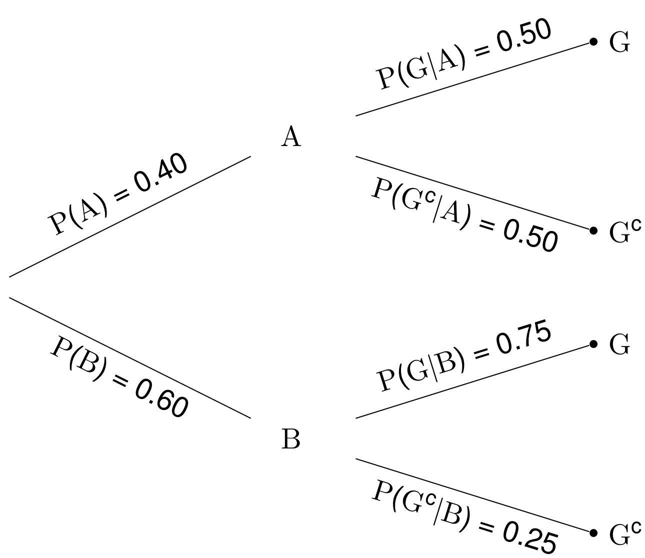
Using the Theorem of Total Probability, the unconditional probability that a randomly selected batch of seeds will be germinated is:

$$\mathrm{P}(G) = \mathrm{P}(G|A)\mathrm{P}(A) + \mathrm{P}(G|B)\mathrm{P}(B) = 0.50 * 0.40 + 0.75 * 0.60 = 0.65.$$

Now, applying Bayes' Theorem, we finally obtain

$$\mathrm{P}(A|G) = \frac{\mathrm{P}(A)\mathrm{P}(G|A)}{\mathrm{P}(G)} = \frac{0.50 * 0.40}{0.65} \approx 0.30769.$$

Alternative Solution: You can make use of a **Tree Diagram** in this case as follows.



Observe that, $G = (G \cap A) \cup (G \cap B)$, and the events $G \cap A$ and $G \cap B$ are disjoint. Hence, using the addition law coupled with the multiplication law, we obtain

$$\begin{aligned}
\mathrm{P}(G) &= \mathrm{P}\left((G \cap A) \cup (G \cap B)\right) \\
&= \mathrm{P}(G \cap A) + \mathrm{P}(G \cap B) \\
&= \mathrm{P}(G|A)\mathrm{P}(A) + \mathrm{P}(G|B)\mathrm{P}(B) \\
&= 0.50 * 0.40 + 0.75 * 0.60 \\
&= 0.65.
\end{aligned}$$

Now, applying the definition of conditional probability together with a straightforward application of the Multiplication Law to the numerator we obtain:

$$\mathrm{P}(A|G) = \frac{\mathrm{P}(A \cap G)}{\mathrm{P}(G)} = \frac{\mathrm{P}(A)\mathrm{P}(G|A)}{\mathrm{P}(G)} = \frac{0.50 * 0.40}{0.65} \approx 0.30769.$$

16. The 'prostate specific antigen' test (PSA) tests for prostate cancer. Suppose that the probability the test says you have the disease given you actually have the disease is 0.99. This is usually referred to as a "true positive". Also, the probability the test says you have the disease given you don't have the disease is 0.02. This is usually referred to as a "false positive". The total incidence rate of prostate cancer in the U.S. male population is 0.6%.

(a) What is the probability that a randomly selected American male will be tested positive?

Solution: Let D and T denote the events that a randomly selected US male have prostate cancer and that a randomly selected US male have been tested positive, respectively.

According to the problem, $\mathrm{P}(\mathrm{D}) = 0.006$, $\mathrm{P}(\mathrm{T}|\mathrm{D}) = 0.99$, and $\mathrm{P}(\mathrm{T}|\mathrm{D}^c) = 0.02$.

This implies that

$$\begin{aligned} \mathrm{P}(\mathrm{D}^c) &= 1 - \mathrm{P}(\mathrm{D}) = 1 - 0.006 = 0.994 \\ \mathrm{P}(\mathrm{T}^c|\mathrm{D}) &= 1 - \mathrm{P}(\mathrm{T}|\mathrm{D}) = 1 - 0.99 = 0.01 \\ \mathrm{P}(\mathrm{T}^c|\mathrm{D}^c) &= 1 - \mathrm{P}(\mathrm{T}|\mathrm{D}^c) = 1 - 0.02 = 0.98. \end{aligned}$$

Using the Theorem of Total Probability, the unconditional probability that a randomly selected American male will be tested positive is:

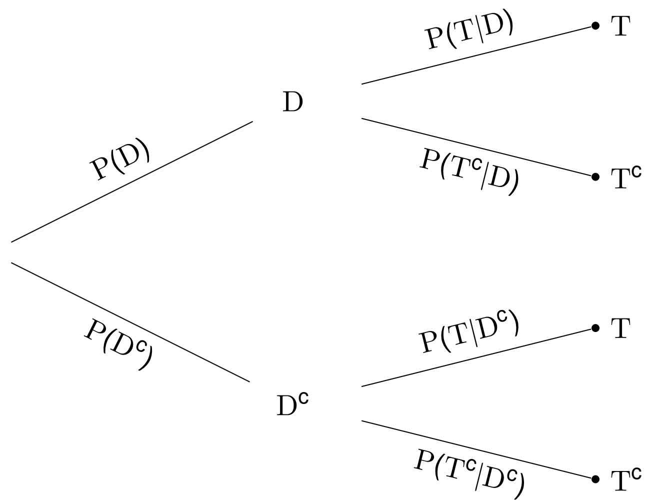
$$\mathrm{P}(T) = \mathrm{P}(T|D)\mathrm{P}(D) + \mathrm{P}(T|D^c)\mathrm{P}(D^c) = 0.99 * 0.006 + 0.02 * 994 = 0.02582.$$

(b) What is the probability that a randomly selected American male has prostrate cancer given that he has been tested positive?

Solution: Using Bayes' Theorem, we obtain

$$\mathrm{P}(D|T) = \frac{\mathrm{P}(T|D)\mathrm{P}(D)}{\mathrm{P}(T)} = \frac{0.99 * 0.006}{0.02582} \approx 0.23.$$

Alternative Solution: You can make use of a **Tree Diagram** in this case as follows.



Observe that, T = $(T \cap D) \cup (T \cap D^c)$, and the events $T \cap D$ and $T \cap D^c$ are disjoint. Hence, using the addition law coupled with the multiplication law, we obtain

$$\begin{aligned} P(T) &= P\left((T \cap D) \cup (T \cap D^c)\right) \\ &= P(T \cap D) + P(T \cap D^c) \\ &= P(T|D)P(D) + P(T|D^c)P(D^c) \\ &= 0.99 * 0.006 + 0.02 * 0.994 \\ &= 0.02582. \end{aligned}$$

Now, applying the definition of conditional probability together with a straightforward application of the Multiplication Law to the numerator we obtain:

$$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{P(A)P(G|A)}{P(G)} = \frac{0.99 * 0.006}{0.02582} \approx 0.23.$$