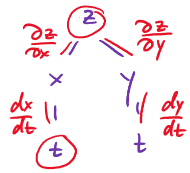
1 Math 251/221 WEEK in REVIEW 5. Fall 2024

1.1 Sections 14.5 - 14.8.

1. If $z = \dfrac{y}{y+x^2}$, $x = \sqrt{t}$ and $y = \ln(t)$, find $\dfrac{dz}{dt}$.



$$\frac{dx}{dt} = (\sqrt{t})' = \frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt} = (\ln t)' = \frac{1}{t}$$

$$\frac{\partial z}{\partial x} = -\frac{y}{(y+x^2)^2}(2x) = -\frac{2xy}{(y+x^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1(y+x^2) - 1(y)}{(y+x^2)^2} = \frac{x^2}{(y+x^2)^2}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= -\frac{y}{(y+x^2)^2}(x^2)'\frac{1}{2\sqrt{t}} + \frac{y+x^2-y}{(y+x^2)^2}\frac{1}{t}$$

$$= -\frac{\cancel{2}xy}{(y+x^2)^2}\frac{1}{\cancel{2}\sqrt{t}} + \frac{x^2}{(y+x^2)^2}\cdot\frac{1}{t}$$

2. Let

$$w = \cos xy + y \cos x,$$

where

$x(0,4) = 1 + 12 = 13$, $y(0,4) = 5 - 2 = 3$

$$x = e^{-t} + 3s,\ y = 5e^{2t} - \sqrt{s}$$

Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$. (evaluate them when $t=0,\ s=4$)

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(-y \sin xy - y \sin x\right)\left(-e^{-t}\right) + \left(-x \sin xy + \cos x\right)\left(10e^{2t}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

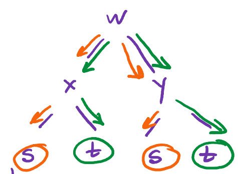
$$= -y\left(\sin xy + \sin x\right)(3) + \left(\cos x - x \sin xy\right)\left(-\frac{1}{2\sqrt{s}}\right)$$

$$\frac{\partial w}{\partial x} = -\sin xy\ \underbrace{(xy)'_x}_{\text{derivative for } x} - y \sin x$$

$$= -y \sin xy - y \sin x$$

$$\frac{\partial w}{\partial y} = -\sin(xy)\,(xy)'_y + \cos x$$

$$= -x \sin xy + \cos x$$



$$\frac{\partial x}{\partial s} = 3,\quad \frac{\partial y}{\partial s} = -\frac{1}{2\sqrt{s}}$$

$$\frac{\partial x}{\partial t} = -e^{-t},\quad \frac{\partial y}{\partial t} = 10e^{2t}$$

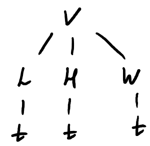
3. The dimensions of a closed box are L, W and H. At a certain instant the dimensions are $L = 1$m and $W = H = 2$m, and L and W are increasing ar a rate of 2 m/s while H is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantites are changing.

   (a) The volume
   (b) The surface area
   (c) The length of a diagonal

$\frac{dL}{dt} = \frac{dW}{dt} = 2, \quad \frac{dH}{dt} = -3$

$L = 1, \quad W = H = 2$

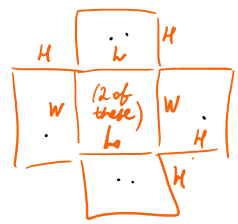
(a) $V = LWH$, $\frac{dV}{dt} = ?$



$$\frac{dV}{dt} = \frac{\partial V}{\partial L}\frac{dL}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt} + \frac{\partial V}{\partial W}\frac{dW}{dt}$$

$$= WH\frac{dL}{dt} + LW\frac{dH}{dt} + LH\frac{dW}{dt}$$

$$= 4(2) + 2(-3) + 2(2) = 6\ (m^3/s)$$

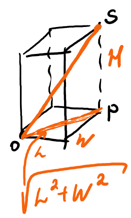


(b). $A = S.A. = 2WH + 2LH + 2LW$

$$\frac{dA}{dt} = \frac{\partial A}{\partial L}\frac{dL}{dt} + \frac{\partial A}{\partial H}\frac{dH}{dt} + \frac{\partial A}{\partial W}\frac{dW}{dt}$$

$$= (2H + 2W)\frac{dL}{dt} + (2W + 2L)\frac{dH}{dt} + (2H + 2L)\frac{dW}{dt}$$

$$= (8)(2) + 6(-3) + 6(2) = 10\ (m^2/s)$$



$OP^2 = L^2 + W^2$

$D^2 = OS^2 = OP^2 + SP^2$
$= L^2 + W^2 + H^2$

$L = 1, \ H = W = 2.$
$D = \sqrt{1+4+4} = \sqrt{9} = 3$

The diagonal $\frac{d}{dt}(D^2) = \frac{d}{dt}(L^2 + H^2 + W^2)$

$$\cancel{2}D\frac{dD}{dt} = \cancel{2}L\frac{dL}{dt} + \cancel{2}H\frac{dH}{dt} + \cancel{2}W\frac{dW}{dt}$$

$$\frac{dD}{dt} = \frac{1}{D}\left(L\frac{dL}{dt} + H\frac{dH}{dt} + W\frac{dW}{dt}\right)$$

$$= \frac{1}{3}\left(1 \cdot 2 + 2(-3) + 2 \cdot (2)\right)$$

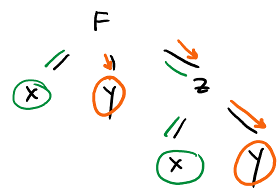
$$= \frac{1}{3}(0) = 0.$$

4. If

$$yz^4 + xz^3 = e^{xyz}$$

find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$F(x,y,z) = 0$



$$\frac{\partial}{\partial x}\big(F(x,y,z)\big) = \frac{\partial}{\partial x}(0)$$

$$0 = \frac{\partial F}{\partial x}\cancelto{1}{\frac{dx}{dx}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}}$$

$$\frac{\partial}{\partial y}\big(F(x,y,z)\big) = \frac{\partial}{\partial y}(0)$$

$$\frac{\partial F}{\partial y}\cancelto{1}{\frac{dy}{dy}} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

$$F(x,y,z) = yz^4 + xz^3 - e^{xyz}$$

$$\frac{\partial F}{\partial x} = F_x = z^3 - e^{xyz}(yz)$$

$$\frac{\partial F}{\partial y} = F_y = z^4 - e^{xyz}(xz)$$

$$\frac{\partial F}{\partial z} = F_z = 4yz^3 + 3xz^2 - e^{xyz}(xy)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z^3 - yz\,e^{xyz}}{4yz^3 + 3xz^2 - xy\,e^{xyz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z^4 - xz\,e^{xyz}}{4yz^3 + 3xz^2 - xy\,e^{xyz}}$$

5. Let

$$f(x,y,z) = x^2y + x\sqrt{1+z}$$

(a) Find $\nabla f(1,2,3)$, the gradient of the function at $(1,2,3)$.

(b) Find $D_{\mathbf{v}} f(1,2,3)$, the directional derivative of f at $(1,2,3)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

$\nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle 2xy + \sqrt{1+z},\ x^2,\ \frac{x}{2\sqrt{1+z}} \rangle$

(a) $\nabla f(1,2,3) = \langle 2(1)(2) + \sqrt{3+1},\ 1^2,\ \frac{1}{2\sqrt{1+3}} \rangle$

$= \langle 6, 1, \frac{1}{4} \rangle$

(b) $\vec{v} = \langle 2, 1, -2 \rangle$, $|\vec{v}| = \sqrt{4+1+4} = 3$

Find a unit vector in the direction of $\vec{v}$

$\vec{u} = \frac{\langle 2, 1, -2 \rangle}{3}$

$D_{\vec{v}} f = \vec{u} \cdot \nabla f(1,2,3) = \langle 6, 1, \frac{1}{4} \rangle \cdot \frac{\langle 2, 1, -2 \rangle}{3}$

$= \frac{12 + 1 - \frac{1}{2}}{3} = \dots$

6. For the function $f(x, y, z) = ze^{xy}$ find

(a) the direction at which it increases most rapidly at the point $(0, 1, 2)$.

(b) What is the maximum rate of increase?

(c) What is the largest rate of decrease of f at this point? In which direction does this change occur?

(d) When is the directional derivative at this point is half of its maximum value?

(a) $\nabla f(0,1,2)$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle ze^{xy}(y), ze^{xy}(x), e^{xy} \rangle$

$\nabla f(0,1,2) = \langle 2e^0, 0, e^0 \rangle = \boxed{\langle 2, 0, 1 \rangle}$

(b) max rate of increase is $|\langle 2, 0, 1 \rangle| = \sqrt{4+1} = \sqrt{5}$

(c) $-\sqrt{5} = -|\langle 2, 0, 1 \rangle|$

in the direction of $-\nabla f(0,1,2)$

(d) $\vec{u}$, $|\vec{u}| = 1$.

$|\nabla f(0,1,2) \cdot \vec{u}| = \frac{\sqrt{5}}{2}$

$\nabla f(0,1,2) \cdot \vec{u} = |\nabla f(0,1,2)| \cdot |\vec{u}| \cos\theta \Rightarrow \theta = \frac{\pi}{3}$.

at any unit vector that would make an angle of $\frac{\pi}{3}$ with $\nabla f(1,2,3)$

7. (a) Find parametric equations of the normal line and an equation of the tangent plane to the surface

$$x^3 + y^3 + z^3 = 5xyz$$

at the point $(2, 1, 1)$.

$F(x,y,z) = x^3 + y^3 + z^3 - 5xyz$

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 3x^2 - 5yz, 3y^2 - 5xz, 3z^2 - 5xy \rangle$

$\nabla F(2,1,1) = \langle 3(4) - 5, 3 - 5(2), 3 - 5(2) \rangle$

$= \langle 7, -7, -7 \rangle$

$= 7 \langle 1, -1, -1 \rangle$ use as a normal vector

tangent plane: $1(x-2) - 1(y-1) - 1(z-1) = 0$

normal line: $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-1}{-1}$ or $\begin{cases} x = 2 + t \\ y = 1 - t \\ z = 1 - t \end{cases}$

8. Find the local maximum and local minimum values, and saddle points if any, of the function $f(x,y) = x^2 - y^2 + xy$

$f(x,y) = x^2 - y^2 + xy$

$\nabla f = \langle 2x+y, -2y+x \rangle = \langle 0,0 \rangle$

$\begin{cases} 2x+y=0 \\ -2y+x=0 \end{cases} \Rightarrow x=2y$ (plug into the first eqn).

$2(2y)+y=0$ or $4y+y=0$ or $5y=0$, $\boxed{y=0}$

$x=2y \rightarrow x=2(0)=0$

$\boxed{(0,0)}$ - critical point.

2nd derivative test:

$D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$

$f_{xx} = 2$, $f_{xy} = 1$, $f_{yy} = -2$

$D = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4-1=-5<0$

(0,0) is a saddle point.

$x>0,\ y>0,\ z>0$

9. Find the three positive numbers x, y, and z whose sum is 100 such that $xy\ z$ is a maximum.

$f(x,y,z) = xyz \leftarrow$ maximize

$x+y+z = 100 \Rightarrow z = 100 - x - y$

$f(x,y) = xy(100-x-y) = 100xy - x^2y - xy^2$

$\nabla f = \langle f_x, f_y \rangle$

$= \langle 100y - 2xy - y^2,\ 100x - x^2 - 2xy \rangle = \langle 0, 0 \rangle$

$\begin{cases} 100y - 2xy - y^2 = 0 \\ 100x - x^2 - 2xy = 0 \end{cases} \Rightarrow y(100 - 2x - y) = 0$

since $y \neq 0$, then $2x + y = 100$

$y = 100 - 2x$ (plug into the 2nd eqn).

$x(100 - x - 2y) = 0$

since $x \neq 0$, then $100 - x - 2y = 0$

$y = 100 - 2x$

$100 - x - 2(100 - 2x) = 0$

$100 - x - 200 + 4x = 0$ or $3x = 100$, $\boxed{x = \frac{100}{3}}$

$y = 100 - 2x = 100 - \frac{200}{3} = \frac{100}{3}$

$z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

$\boxed{x = y = z = \frac{100}{3}}$

2nd derivative test.

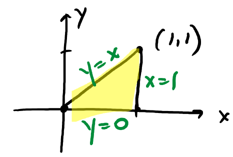
$f_{xx} = -2y,\ f_{xy} = 100 - 2x - 2y,\ f_{yy} = -2x$

$f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$, $f_{xy} = 100 - \frac{200}{3} - \frac{200}{3} = -\frac{100}{3}$, $f_{yy}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3}$

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -\frac{200}{3} & -\frac{100}{3} \\ -\frac{100}{3} & -\frac{200}{3} \end{vmatrix} = \frac{40000}{9} - \frac{10000}{9} = \frac{30000}{9} > 0$

$D > 0$, $f_{xx} < 0$, $\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local max.

10. Find the absolute maximum and minimum values of $f(x,y) = 3x^2y - x^3 - y^4$ on the closed triangular region in the xy-plane with the vertices $(0,0)$, $(1,1)$, and $(1,0)$.



1. Find critical points inside D
2. Find critical points on the boundary
3. Evaluate $f(x,y)$ at the points from steps 1 and 2.
   The smallest number is the abs min value
   The largest number is the abs max value.

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1. Inside D

$\nabla f = \langle f_x, f_y \rangle = \langle 6xy - 3x^2,\ 3x^2 - 4y^3 \rangle = \langle 0, 0 \rangle$

$\begin{cases} 6xy - 3x^2 = 0 \\ 3x^2 - 4y^3 = 0 \end{cases} \longrightarrow 3x(2y-x) = 0$

$x=0$ or $2y - x = 0$

If $x=0$, then $-4y^3 = 0 \Rightarrow y=0 \Rightarrow \boxed{(0,0)}$

If $x = 2y \Rightarrow 3(2y)^2 - 4y^3 = 0$

$12y^2 - 4y^3 = 0$

$4y^2(3-y) = 0$

$y=0$ or $y=3$

$x = 2(0) = 0$, $x = 2(3) = 6$

$\boxed{(0,0)}$ $(6,3) \leftarrow$ out the region D

The boundaries.

(1) $y=0$, $0 \le x \le 1$

$f(x,0) = -x^3$

$f'(x,0) = -3x^2 = 0 \Rightarrow x = 0 \Rightarrow \boxed{(0,0)}$

(2) $y = x$

$f(x,x) = 3x^3 - x^3 - x^4 = 2x^3 - x^4$

$f'(x,x) = 6x^2 - 4x^3 = 0 \Rightarrow 2x^2(3 - 2x) = 0$

$x = 0$, $3 - 2x = 0 \Rightarrow x = \frac{3}{2}$

$\boxed{(0,0)}$ $\left(\frac{3}{2}, \frac{3}{2}\right)$ not on the region.

(3) $x=1$, $0 \le y \le 1$

$f(1,y) = 3y - 1 - y^4$

$f'(1,y) = 3 - 4y^3 = 0 \Rightarrow y^3 = \frac{3}{4} \Rightarrow y = \sqrt[3]{\frac{3}{4}}$

$\boxed{\left(1, \sqrt[3]{\frac{3}{4}}\right)}$

$f(0,0) = \boxed{0}$ abs min.

$f\left(1, \sqrt[3]{\frac{3}{4}}\right) = 3 \cdot \sqrt[3]{\frac{3}{4}} - 1 - \left(\sqrt[3]{\frac{3}{4}}\right)^4 = \frac{11}{4} \cdot \sqrt[3]{\frac{3}{4}} - 1 \approx \boxed{1.5}$ abs. max

11. Use Lagrange multipliers to find the maximum and minimum values of the function $\underbrace{f(x,y) = xy}$ subject to the constraint $\underbrace{9x^2 + y^2 = 4}$.

$g(x,y) = 9x^2+y^2-4$

Looking for points (x,y) where $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$

$f(x,y) = xy$

$\nabla f = \langle y, x \rangle$

$\nabla g = \langle 18x, 2y \rangle$

$\Rightarrow \begin{cases} y = 18\lambda x \Rightarrow \lambda = \dfrac{y}{18x} \\ x = 2\lambda y \\ 9x^2 + y^2 = 4 \end{cases}$

$x = 2y \dfrac{y}{18x}$

$x = \dfrac{y^2}{9x}$ or $9x^2 = y^2$

$9x^2 - y^2 = 0$

$(3x-y)(3x+y) = 0 \Rightarrow y = 3x$ or $y = -3x$

$y = 3x \longrightarrow 9x^2+y^2=4$

$9x^2+9x^2=4$

$18x^2=4$ or $x^2 = \frac{2}{9}$, $x = \pm\frac{\sqrt{2}}{3}$, $y = 3x = \pm\sqrt{2}$

$\boxed{\left(\frac{\sqrt{2}}{3}, \sqrt{2}\right), \left(-\frac{\sqrt{2}}{3}, -\sqrt{2}\right)}$

$y = -3x \longrightarrow 9x^2 + y^2 = 4$

$18x^2 = 4$ or $x = \pm\frac{\sqrt{2}}{3} \Rightarrow y = \mp\sqrt{2}$

$\boxed{\left(\frac{\sqrt{2}}{3}, -\sqrt{2}\right), \left(-\frac{\sqrt{2}}{3}, \sqrt{2}\right)}$

$f(x,y) = xy$

$f\left(\frac{\sqrt{2}}{3}, \sqrt{2}\right) = \frac{2}{3}$

$f\left(-\frac{\sqrt{2}}{3}, -\sqrt{2}\right) = \frac{2}{3}$

abs max value

$f\left(\frac{\sqrt{2}}{3}, -\sqrt{2}\right) = -\frac{2}{3}$

$f\left(-\frac{\sqrt{2}}{3}, \sqrt{2}\right) = -\frac{2}{3}$

abs min value

12. Use Lagrange multipliers to find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 9y + 4z = 27$.

$V = xyz$, constraint $x + 9y + 4z = 27$ or $g(x, y, z) = x + 9y + 4z - 27$

$\nabla V = \langle V_x, V_y, V_z \rangle = \langle yz, xz, xy \rangle$

$\nabla g = \langle 1, 9, 4 \rangle = \langle g_x, g_y, g_z \rangle$

$\nabla V = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \lambda \langle 1, 9, 4 \rangle$

$$\begin{cases} yz = \lambda \\ xz = 9\lambda \Rightarrow xz = 9(yz) \Rightarrow x = 9y \Rightarrow \\ xy = 4\lambda \Rightarrow xy = 4(yz) \Rightarrow x = 4z \Rightarrow \\ x + 9y + 4z = 27 \end{cases} \quad \boxed{\begin{matrix} y = \frac{x}{9} \\ z = \frac{x}{4} \end{matrix}}$$

plug into the constraint

$x + 9\frac{x}{9} + 4\frac{x}{4} = 27 \Rightarrow 3x = 27 \Rightarrow x = 9$

$y = 1$

$z = \frac{9}{4}$

the volume $\boxed{V = 9(1)\left(\frac{9}{4}\right) = \frac{81}{4}}$