$$\iint_S f(x,y,z)\,dS, \quad dS = |\vec{n}|\,dA \quad = \iint_D f(\text{parametrization})\,|\vec{n}|\,dA.$$

$S: \ \vec{r}(u,v), \quad \vec{n} = \pm \vec{r}_u \times \vec{r}_v$

$S: \ z = z(x,y) \Rightarrow \vec{n} = \pm \langle z_x, z_y, -1 \rangle$

1. Evaluate the surface integral $\displaystyle\iint_S (x+y+z)\,dS$, where S is the parallelogram with parametric equations $x = u+v$, $y = 1-v$, $z = 1+2u+v$, $0 \le u \le 2$, $0 \le v \le 1$.

$S: \ \vec{r}(u,v) = \langle u+v, 1-v, 1+2u+v \rangle$

$\vec{n} = \vec{r}_u \times \vec{r}_v$

$\vec{r}_u = \langle 1, 0, 2 \rangle$

$\vec{r}_v = \langle 1, -1, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 2\vec{i} - \vec{j}(1-2) - \vec{k}$$

$$= 2\vec{i} + \vec{j} - \vec{k}$$

$|\vec{n}| = \sqrt{4+1+1} = \sqrt{6}$

$$\iint_S (x+y+z)\,dS \ \left| \begin{array}{l} x = u+v \\ y = 1-v \\ z = 1+2u+v \\ dS = \sqrt{6}\,dA \\ 0 \le u \le 2 \\ 0 \le v \le 1 \end{array} \right| = \iint_{\substack{0\le u\le 2 \\ 0 \le v \le 1}} [\underbrace{u+v}_{x} + \underbrace{1-v}_{y} + \underbrace{1+2u+v}_{z}]\underbrace{\sqrt{6}\,dA}_{dS}$$

$$= \sqrt{6}\int_0^2 \int_0^1 [3u + v + 2]\,dv\,du = \sqrt{6}\int_0^2 \left(3uv + \frac{v^2}{2} + 2v\right)\Big|_0^1 du = \sqrt{6}\int_0^2 \left(3u + \frac{1}{2} + 2\right) du$$

$$= \sqrt{6}\int_0^2 \left(3u + \frac{5}{2}\right) du = \sqrt{6}\left(\frac{3u^2}{2} + \frac{5}{2}u\right)\Big|_0^2 = 11\sqrt{6}$$

2. Evaluate the surface integral $\iint\limits_S (x^2 z + y^2 z)dS$ where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $\underbrace{x^2 + y^2 = 4}_{\text{parameter domain.}}$.

$z = 4+x+y$, $\quad \vec{n} = \langle -z_x, -z_y, 1\rangle = \langle -1, -1, 1\rangle$

$|\vec{n}| = \sqrt{1+1+1} = \sqrt{3}$

$$\iint\limits_S (x^2+y^2)z\,dS \quad \left| \begin{array}{l} z = 4+x+y \\ dS = \sqrt{3}\,dA \end{array} \right| = \iint\limits_{x^2+y^2\le 4} (x^2+y^2)(4+x+y)\sqrt{3}\,dA \quad \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \\ 0\le r\le 2 \\ 0\le\theta\le 2\pi \end{array} \right|$$

$$= \sqrt{3}\int_0^{2\pi}\int_0^2 r^2(4 + r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \sqrt{3}\int_0^{2\pi}\int_0^2 \left[4r^3 + r^4\cos\theta + r^4\sin\theta\right] dr\,d\theta$$

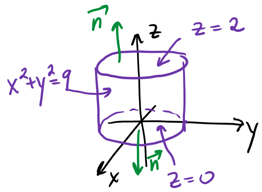
$$= \sqrt{3}\int_0^{2\pi} \left(\frac{4r^4}{4} + \frac{r^5}{5}\cos\theta + \frac{r^5}{5}\sin\theta\right)\Big|_0^2 \, d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left(16 + \frac{32}{5}\cos\theta + \frac{32}{5}\sin\theta\right) d\theta$$

$$= \sqrt{3}\left(16\theta + \frac{32}{5}\sin\theta - \frac{32}{5}\cos\theta\right)\Big|_0^{2\pi}$$

$$= \boxed{32\sqrt{3}\pi}$$

3. Evaluate $\displaystyle\iint_S (x^2+y^2+z^2)\,dS$, S is the part of the cylinder $x^2+y^2=9$ between the planes $z=0$ and $z=2$, together with its top and bottom disks.



$$\iint_S (x^2+y^2+z^2)\,dS = \iint_{z=0} (x^2+y^2+z^2)\,dS + \iint_{z=2} (x^2+y^2+z^2)\,dS + \iint_{x^2+y^2=9} (x^2+y^2+z^2)\,dS$$

$\boxed{z=0}$ $\quad \vec n = -\vec k = -\langle 0,0,1\rangle$

$|\vec n| = 1$

$|\vec n|\,dA = dS$

$$\iint_{z=0} (x^2+y^2+z^2)\,dS = \iint_{x^2+y^2\le 9} (x^2+y^2+0^2)\,dA = \iint_{x^2+y^2\le 9} (x^2+y^2)\,dA = \left|\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \\ 0\le\theta\le 2\pi \\ 0\le r\le 3 \end{array}\right|$$

$$= \int_0^{2\pi}\int_0^3 r^2\, r\,dr\,d\theta = 2\pi \left.\frac{r^4}{4}\right|_0^3 = \boxed{\frac{81}{2}\pi}$$

$\boxed{z=2}$ $\quad \vec n = \vec k = \langle 0,0,1\rangle$, $\quad |\vec n| = 1$

$$\iint_{z=2} (x^2+y^2+z^2)\,dS = \iint_{x^2+y^2\le 9} (x^2+y^2+2^2)\,dA = \iint_{x^2+y^2\le 9} (x^2+y^2+4)\,dA \quad \left|\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \end{array}\right|$$

$$= \int_0^{2\pi}\int_0^3 (r^2+4)\,r\,dr\,d\theta = 2\pi\int_0^3 (r^3+4r)\,dr = 2\pi\left(\frac{r^4}{4}+\frac{4r^2}{2}\right)\Big|_0^3$$

$$= 2\pi\left(\frac{81}{4}+18\right) = \pi\left(\frac{81}{2}+36\right) = \boxed{\frac{153\pi}{2}}$$

$\boxed{x^2+y^2=9}$

$$\begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \\ z = z \end{cases} \Rightarrow \vec r(\theta,z) = \langle 3\cos\theta,\ 3\sin\theta,\ z\rangle$$

$0\le\theta\le 2\pi$

$0\le z\le 2$

$\vec n = \vec r_\theta \times \vec r_z$

$\vec r_\theta = \langle -3\sin\theta,\ 3\cos\theta,\ 0\rangle$

$\vec r_z = \langle 0,\ 0,\ 1\rangle$

$$\vec n = \vec r_\theta\times\vec r_z = \begin{vmatrix} \vec i & \vec j & \vec k \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec i\begin{vmatrix} 3\cos\theta & 0 \\ 0 & 1\end{vmatrix} - \vec j\begin{vmatrix} -3\sin\theta & 0 \\ 0 & 1\end{vmatrix} + \vec k\begin{vmatrix} -3\sin\theta & 3\cos\theta \\ 0 & 0\end{vmatrix}$$

$$= 3\cos\theta\,\vec i + 3\sin\theta\,\vec j$$

$$|\vec n| = \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3$$

$$\iint_{x^2+y^2=9} (x^2+y^2+z^2)\,dS = \left|\begin{array}{l} x = 3\cos\theta \\ y = 3\sin\theta \\ z = z \\ dS = |\vec n|\,dA = 3\,dA \\ 0\le\theta\le 2\pi \\ 0\le z\le 2 \end{array}\right| = \int_0^{2\pi}\int_0^2 (9+z^2)\,3\,dz\,d\theta$$

$$= 6\pi\left(9z+\frac{z^3}{3}\right)\Big|_0^2$$

$$= 6\pi\left(18+\frac{8}{3}\right) = \frac{372\pi}{3} = \boxed{124\pi}$$

$$\iint_S (x^2+y^2+z^2)\,dS = 124\pi + \frac{153\pi}{2} + \frac{81\pi}{2} = \boxed{241\pi}$$

$\vec{n} = \pm \vec{r}_u \times \vec{r}_v$,

$z = z(x,y)$, $\vec{n} = \pm \langle z_x, z_y, -1 \rangle$

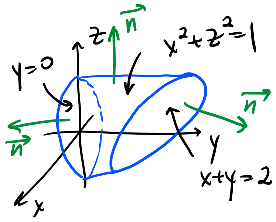
$x = x(y,z)$, $\vec{n} = \pm \langle -1, x_y, x_z \rangle$

$y = y(x,z)$, $\vec{n} = \pm \langle y_x, -1, y_z \rangle$

4. Find flux of the vector field $\mathbf{F} = \langle x, y, 5 \rangle$ across the surface S which is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$. Use positive (outward) orientation for S.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\text{parametrization}) \cdot \vec{n}\, dA$$

$$\iint_S \vec{F} \cdot d\vec{S} = \underbrace{\iint_{y=0} \vec{F} \cdot d\vec{S}}_{0} + \underbrace{\iint_{x+y=2} \vec{F} \cdot d\vec{S}}_{2\pi} + \underbrace{\iint_{x^2+z^2=1} \vec{F} \cdot d\vec{S}}_{2\pi} = \boxed{4\pi}$$



$\boxed{y=0}$ $\vec{n} = -\vec{j} = -\langle 0,1,0 \rangle = \langle 0,-1,0 \rangle$

$\vec{F} = \langle x, y, 5 \rangle \xrightarrow{y=0} \langle x, 0, 5 \rangle$

$\vec{F} \cdot \vec{n} = \langle x,0,5 \rangle \cdot \langle 0,-1,0 \rangle = \boxed{0}$

must be positive

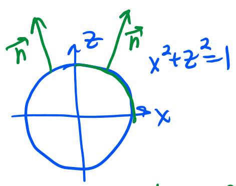
$\boxed{x+y=2}$ $y = 2-x$, $\vec{n} = \pm \langle y_x, -1, y_z \rangle = \pm \langle -1, -1, 0 \rangle = \langle 1,1,0 \rangle$

parameter domain is $x^2+z^2 \le 1$

$\vec{F} = \langle x, y, 5 \rangle \xrightarrow{y=2-x} \langle x, 2-x, 5 \rangle$

$\vec{F} \cdot \vec{n} = \langle x, 2-x, 5 \rangle \cdot \langle 1,1,0 \rangle = x + 2 - x = 2$

$$\iint_{x+y=2} \vec{F} \cdot d\vec{S} = \iint_{x^2+z^2 \le 1} 2\, dA = 2 \iint_{x^2+z^2 \le 1} dA = \boxed{2\pi}$$



1st quadrant: $\cos\theta \ge 0$
$\sin\theta \ge 0$.

the component of $\vec{n}$ are positive , when $0 \le \theta \le \pi/2$

$\boxed{x^2+z^2=1}$

$$\begin{cases} x = \cos\theta \\ y = y \\ z = \sin\theta \end{cases}$$

$$\boxed{\begin{array}{l} 0 \le \theta \le \pi \\ 0 \le y \le 2-x \\ 0 \le y \le 2-\cos\theta \end{array}}$$ parameter domain.

$\vec{n} = \pm \vec{r}_\theta \times \vec{r}_y$

$\vec{r}(\theta, y) = \langle \cos\theta, y, \sin\theta \rangle$

$\vec{r}_\theta = \langle -\sin\theta, 0, \cos\theta \rangle$

$\vec{r}_y = \langle 0, 1, 0 \rangle$

$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & \cos\theta \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -\cos\theta\, \vec{i} - \sin\theta\, \vec{k}$$

$\vec{n} = \pm \langle -\cos\theta, 0, -\sin\theta \rangle = \langle \cos\theta, 0, \sin\theta \rangle$

$\vec{F} = \langle x, y, 5 \rangle = \langle \cos\theta, y, 5 \rangle$

$\vec{F} \cdot \vec{n} = \langle \cos\theta, y, 5 \rangle \cdot \langle \cos\theta, 0, \sin\theta \rangle$

$$\vec{F} = \langle x, y, 5 \rangle = \langle \cos\theta, y, 5 \rangle$$

$$\vec{F}\cdot\vec{n} = \langle \cos\theta, y, 5\rangle \cdot \langle \cos\theta, 0, \sin\theta \rangle$$

$$= \cos^2\theta + 5\sin\theta$$

$$\iint_{x^2+z^2=1} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{2-\cos\theta} \left(\cos^2\theta + 5\sin\theta\right) dy\, d\theta$$

$$= \int_0^{2\pi} \left(\cos^2\theta + 5\sin\theta\right) y \Big|_0^{2-\cos\theta} d\theta$$

$$= \int_0^{2\pi} \left(\cos^2\theta + 5\sin\theta\right)\left(2 - \cos\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(2\cos^2\theta + 10\sin\theta - \cos^3\theta - 5\cos\theta\sin\theta\right) d\theta$$

$$= 2\int_0^{2\pi} \cos^2\theta\, d\theta + 10\int_0^{2\pi}\sin\theta\, d\theta - \int_0^{2\pi} \underbrace{\cos^3\theta}_{\cos^2\theta\cdot\cos\theta}\, d\theta - 5\,\cancel{\int_0^{2\pi}\cos\theta\sin\theta\, d\theta} \to 0$$

$u = \sin\theta$ | $u(0) = \sin 0 = 0$

$du = \cos\theta\, d\theta$ | $u(2\pi) = \sin 2\pi = 0$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \int_0^{2\pi}\left(1 + \cos 2\theta\right) d\theta - 10\,\cancel{\cos\theta\Big|_0^{2\pi}} \to 0 \;-\; \cancel{\int_0^{2\pi}\cos\theta\left(1 - \sin^2\theta\right) d\theta} \to 0$$

$u = \sin\theta$

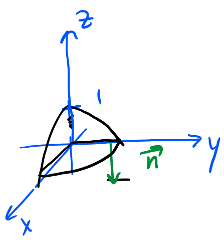
$du = \cos\theta\, d\theta$

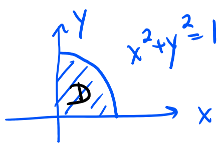
$u(0) = \sin 0 = 0$

$u(2\pi) = \sin 2\pi = 0$

$$= \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \boxed{2\pi}$$

5. Evaluate the surface integral $\iint_S < x, y, 1 > \cdot d\mathbf{S}$ where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ in the first octant, oriented by downward normals.





$z = 1 - x^2 - y^2$

$\vec{n} = \pm < z_x, z_y, -1 > = \oplus < -2x, -2y, -1 > = < -2x, -2y, -1 >$

$< x, y, 1 > \cdot < -2x, -2y, -1 > = -2x^2 - 2y^2 - 1$

$$\iint_S \vec{F} \cdot d\vec{S} = -\iint_D (2x^2 + 2y^2 + 1)\, dA = \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \\ 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{array} \right|$$

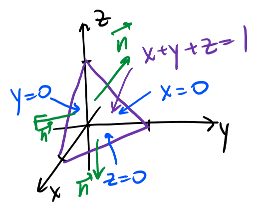
$$= -\int_0^{\pi/2}\int_0^1 (2r^2 + 1)\, r\,dr\,d\theta$$

$$= -\frac{\pi}{2}\int_0^1 (2r^3 + r)\, dr$$

$$= -\frac{\pi}{2}\left(\frac{2r^4}{4} + \frac{r^2}{2}\right)\Bigg|_0^1 = -\frac{\pi}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \boxed{-\frac{\pi}{2}}$$

6. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, if $\mathbf{F} =< y, z-y, x >$, and S if the surface of the tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and (0,0,1).

$$\iint_S \vec{F}\cdot d\vec{S} = \underbrace{\iint_{z=0} \vec{F}\cdot d\vec{S}}_{-1/6} + \underbrace{\iint_{x=0} \vec{F}\cdot d\vec{S}}_{-1/6} + \underbrace{\iint_{y=0} \vec{F}\cdot d\vec{S}}_{-1/6} + \underbrace{\iint_{x+y+z=1} \vec{F}\cdot d\vec{S}}_{1/3} = \frac{1}{3} - \frac{1}{2} = \boxed{-\frac{1}{6}}$$



$\boxed{z=0}$ $\vec{n} = -\vec{k} = <0, 0, -1>$

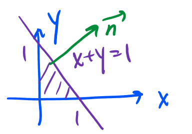
$$\vec{F} = <y, z-y, x> \overset{z=0}{=} <y, -y, x>$$

$$\vec{F}\cdot\vec{n} = <0,0,-1>\cdot<y,-y,x> = -x$$

$$\iint_{z=0} \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-x} (-x)\,dy\,dx = -\int_0^1 x(1-x)\,dx$$

$$= -\int_0^1 (x-x^2)\,dx$$

$$= -\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = -\left(\frac{1}{2}-\frac{1}{3}\right) = \boxed{-\frac{1}{6}}$$



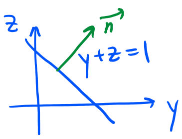
$0 \le y \le 1-x$
$0 \le x \le 1$

$\boxed{x=0}$ $\vec{n} = <-1, 0, 0> = -\vec{i}$

$$\vec{F} = <y, z-y, x> \overset{x=0}{=\!=} <y, z-y, 0>$$

$$\vec{F}\cdot\vec{n} = <y, z-y, 0>\cdot<-1,0,0> = -y$$

$$\iint_{x=0} \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-y} (-y)\,dz\,dy = -\int_0^1 y(1-y)\,dy = -\frac{1}{6}$$



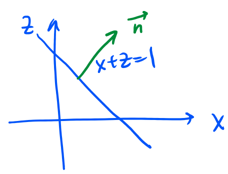
$0 \le z \le 1-y$
$0 \le y \le 1$

$\boxed{y=0}$ $\vec{n} = -\vec{j} = <0, -1, 0>$

$$\vec{F} = <y, z-y, x> \overset{y=0}{=\!=} <0, z, x>$$

$$\vec{F}\cdot\vec{n} = <0,-1,0>\cdot<0,z,x> = -z$$

$$\iint_{y=0} \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-z} (-z)\,dx\,dz = -\frac{1}{6}$$



$0 \le x \le 1-z$
$0 \le z \le 1$

$x+y+z=1 \Rightarrow z = 1-x-y$

$$\vec{n} = \pm <z_x, z_y, -1> = -<-1,-1,-1> = <1,1,1>$$

$$\vec{F} = <y, z-y, x> \overset{z=1-x-y}{=\!=} <y, 1-x-y-y, x>$$

$$= <y, 1-x-2y, x>$$

$$\vec{F}\cdot\vec{n} = <y, 1-x-2y, x>\cdot<1,1,1> = y + 1 - x - 2y + x = 1-y$$

$$\boxed{\text{Parameter domain}: \begin{array}{l} 0 \le x \le 1-y \\ 0 \le y \le 1 \end{array}}$$

$$\iint_{x+y+z=1} \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-y} (1-y)\,dx\,dy = \int_0^1 (1-y)^2\,dy$$

$$= \int_0^1 (1-2y+y^2)\,dy = \left(y - y^2 + \frac{y^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

$x+y+z=1$

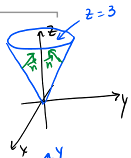
$$= \int_0^1 (1-2y+y^2)\, dy = \left(y - y^2 + \frac{y^3}{3}\right)_0^1 = \frac{1}{3}$$

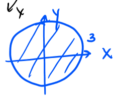
Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z^4\,\mathbf{k}$

S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 3$ with upward orientation

the third component of the normal vector is positive.





$$z = \sqrt{x^2+y^2}$$

$$\vec{n} = \langle -z_x, -z_y, 1\rangle$$

$$\vec{n} = \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right\rangle$$

$$\vec{F} = \langle x, y, z^4\rangle = \left\langle x, y, \left(\sqrt{x^2+y^2}\right)^4\right\rangle = \langle x, y, (x^2+y^2)^2\rangle$$

$$\vec{F}\cdot\vec{n} = \langle x, y, (x^2+y^2)^2\rangle \cdot \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right\rangle$$

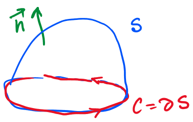
$$= -\frac{x^2+y^2}{\sqrt{x^2+y^2}} + (x^2+y^2)^2 = -\sqrt{x^2+y^2} + (x^2+y^2)^2$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 9} \left(-\sqrt{x^2+y^2} + (x^2+y^2)^2\right) dA = \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \\ 0\le r\le 3 \\ 0\le\theta\le 2\pi \end{array} \right| = \int_0^{2\pi}\int_0^3 (-r + r^4)\, r\,dr\,d\theta$$

$$= 2\pi \int_0^3 (r^5 - r^2)\,dr$$

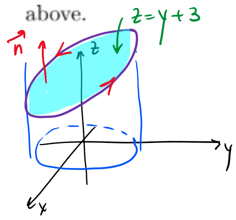
$$= 2\pi\left(\frac{r^6}{6} - \frac{r^3}{3}\right)\Big|_0^3$$

$$= 2\pi\left(\frac{3^6}{6} - \frac{27}{3}\right) = \ldots$$



$$\oint_{C=\partial S} \vec F\cdot d\vec r = \iint_S (\text{curl } \vec F)\cdot d\vec S$$

7. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = < 3z, 5x, -2y >$ and C is the ellipse in which the plane $z = y+3$ intersects the cylinder $x^2+y^2=4$, with positive orientation as viewed from above.



$$\oint_C \vec F\cdot d\vec r = \iint_{z=y+3} (\text{curl } \vec F)\cdot d\vec S$$

$$\text{curl } \vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z & 5x & -2y \end{vmatrix} = \vec i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x & -2y \end{vmatrix} - \vec j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 3z & -2y \end{vmatrix}$$

$$+ \vec k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 3z & 5x \end{vmatrix} = \vec i\left(\overset{-2}{\cancel{\frac{\partial}{\partial y}(-2y)}} - \overset{0}{\cancel{\frac{\partial}{\partial z}(5x)}}\right) - \vec j\left(\overset{0}{\cancel{\frac{\partial}{\partial x}(-2y)}} - \overset{-3}{\cancel{\frac{\partial}{\partial z}(3z)}}\right) + \vec k\left(\overset{5}{\cancel{\frac{\partial}{\partial x}(5x)}} - \overset{0}{\cancel{\frac{\partial}{\partial y}(3z)}}\right)$$

$$\text{curl } \vec F = < -2, 3, 5 >$$

normal vector: $\vec n = < -z_x, -z_y, 1 > \;= < 0, -1, 1 >$

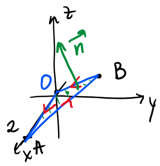
$z = y+3, \quad z_x = 0, \quad z_y = 1$

$$\text{curl } \vec F \cdot \vec n = < -2, 3, 5 > \cdot < 0, -1, 1 > = 2$$

$$\oint_C \vec F\cdot d\vec r = \iint_{z=y+3} \text{curl } \vec F \cdot d\vec S = \iint_{x^2+y^2\le 4} \text{curl } \vec F \cdot \vec n \, dS$$

$$= \iint_{x^2+y^2\le 4} 2\, dA = 2 \iint_{x^2+y^2\le 4} dA = 2\cdot\pi\cdot 2^2 = \boxed{8\pi}$$

8. Find the work performed by the forced field $\mathbf{F} = \langle -3y^2, 4z, 6x \rangle$ on a particle that traverses the triangle C in the plane $z = \frac{1}{2}y$ with vertices $A(2,0,0)$, $B(0,2,1)$, and $O(0,0,0)$ with a counterclockwise orientation looking down the positive z-axis.



$W = \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$

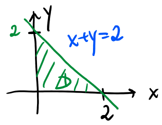
S is the plane that contains $\triangle ABC$.

$\vec{AB} = \langle -2, 2, 1 \rangle$

$\vec{AO} = \langle -2, 0, 0 \rangle$

$$\vec{AB} \times \vec{AO} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 1 \\ -2 & 0 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} -2 & 1 \\ -2 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 2 \\ -2 & 0 \end{vmatrix}$$

$= -2\vec{j} + 4\vec{k}$



$0 \le x \le 2-y$

$0 \le y \le 2$

An equation of the plane with the normal vector $\langle 0, -2, 4 \rangle$ through $(0,0,0)$

$-2y + 4z = 0 \Rightarrow y = 2z \Rightarrow \boxed{z = \frac{y}{2}}$

$\vec{n} = \langle -z_x, -z_y, 1 \rangle = \langle 0, -1/2, 1 \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -3y^2 & 4z & 6x \end{vmatrix} = \vec{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4z & 6x \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ -3y^2 & 6x \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -3y^2 & 4z \end{vmatrix}$$

$$= \vec{i}\left[\overset{0}{\cancel{\frac{\partial}{\partial y}(6x)}} - \overset{4}{\cancel{\frac{\partial}{\partial z}(4z)}}\right] - \vec{j}\left[\overset{6}{\cancel{\frac{\partial}{\partial x}(6x)}} - \overset{0}{\cancel{\frac{\partial}{\partial z}(-3y^2)}}\right] + \vec{k}\left[\overset{0}{\cancel{\frac{\partial}{\partial x}(4z)}} - \overset{-6y}{\cancel{\frac{\partial}{\partial y}(-3y^2)}}\right]$$

$\text{curl } \vec{F} = \langle -4, -6, -6y \rangle$

$\text{curl } \vec{F} \cdot \vec{n} = \langle -4, -6, -6y \rangle \cdot \langle 0, -\frac{1}{2}, 1 \rangle = 3 - 6y$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{z=\frac{y}{2}} \text{curl } \vec{F} \cdot d\vec{S} = \iint_D (3-6y)\, dA = \int_0^2 \int_0^{2-y} (3-6y)\, dx\, dy$$

$$= \int_0^2 (3-6y)(2-y)\, dy = \int_0^2 (6 - 3y - 12y + 6y^2)\, dy$$

$$= \int_0^2 (6 - 15y + 6y^2)\, dy = \left(6y - 15\frac{y^2}{2} + 6\frac{y^3}{3}\right)_0^2$$

$$= 12 - 30 + 16 = \boxed{-2}$$