Math 151 - Week-In-Review 12

Topics for the week:

4.9 Antiderivatives
5.1 Areas and Distances
5.2 The Definite Integral
Review for Exam 3 (3.10 - 5.2)

4.9 Antiderivatives

1. List the antiderivative rules.

$f'(x) = x^n \qquad f(x) = \frac{x^{n+1}}{n+1} + C$

$f'(x) = e^x \qquad f(x) = e^x + C$

$f'(x) = \ln(x) \qquad f(x) = x\ln(x) - x + C$

$f'(x) = \frac{1}{x} \qquad f(x) = \ln|x| + C$

$f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad f(x) = \arcsin(x) + C$

$f'(x) = \frac{1}{1+x^2} \qquad f(x) = \arctan(x) + C$

$f'(x) = \sin(x) \qquad f(x) = -\cos(x) + C$

$f'(x) = \cos(x) \qquad f(x) = \sin(x) + C$

$f'(x) = \sec^2(x) \qquad f(x) = \tan(x) + C$

$f'(x) = \sec(x)\tan(x) \qquad f(x) = \sec(x) + C$

$f'(x) = \csc^2(x) \qquad f(x) = -\cot(x) + C$

$f'(x) = \csc(x)\cot(x) \qquad f(x) = -\csc(x) + C$

2. Compute the most general antiderivative of the function $f'(x) = x^4 + \frac{8}{x} + \frac{3}{x^2} + \sqrt[3]{x} + 7$.

$f'(x) = x^4 + \frac{8}{x} + \frac{3}{x^2} + \sqrt[3]{x} + 7$

$f'(x) = x^4 + 8x^{-1} + 3x^{-2} + x^{1/3} + 7$

$f(x) = \frac{x^{4+1}}{4+1} + 8\ln|x| + \frac{3x^{-2+1}}{-2+1} + \frac{x^{1/3+1}}{1/3+1} + \frac{7x^{0+1}}{1} + C$

$f(x) = \frac{1}{5}x^5 + 8\ln|x| - 3x^{-1} + \frac{3}{4}x^{4/3} + 7x + C$

3. Compute the most general antiderivative of the function $f(x) = 7e^x + \frac{4}{1+x^2} + \sqrt[5]{x^2}$.

$f(x) = 7e^x + \frac{4}{1+x^2} + x^{2/5}$

$F(x) = 7e^x + 4\arctan(x) + \frac{5}{7}x^{7/5} + C$

4. Compute the antiderivative of the function $f'(x) = \sec^2(x) + \sec(x)\tan(x)$ when $f\left(\frac{\pi}{4}\right) = 0$.

$f'(x) = \sec^2(x) + \sec(x)\tan(x)$

$f(x) = \tan(x) + \sec(x) + C$

$f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) + C$

$0 = 1 + \sqrt{2} + C$

$C = -1 - \sqrt{2}$

$f(x) = \tan(x) + \sec(x) - 1 - \sqrt{2}$

5. Compute the position function $s(t)$, given $a(t) = 5\cos(t) - \sin(t)$ and $v(0) = -1$ and $s(0) = 4$.

$v(t) = \int a(t)\, dt = \int (5\cos(t) - \sin(t))\, dt$

$v(t) = 5\sin(t) + \cos(t) + C_1$

$v(0) = 5\sin(0) + \cos(0) + C_1$

$-1 = 5(0) + 1 + C_1$

$-2 = C_1$

$s(t) = \int v(t)\, dt = \int (5\sin(t) + \cos(t) - 2)\, dt$

$s(t) = -5\cos(t) + \sin(t) - 2t + C_2$

$s(0) = -5\cos(0) + \sin(0) - 2(0) + C_2$

$4 = -5(1) + 0 - 0 + C_2$

$9 = C_2$

$s(t) = -5\cos(t) + \sin(t) - 2t + 9$

6. Find the position function of a particle whose movement can be described with the following information.

   $\mathbf{a}(t) = \langle 3\sin(t), 2e^t \rangle$, $\mathbf{v}(0) = \langle 6, 3 \rangle$, $\mathbf{s}(\pi) = \langle 0, \pi \rangle$

$\vec{a}(t) = \langle 3\sin(t), 2e^t \rangle$

$\vec{v}(t) = \langle -3\cos(t) + C_1, 2e^t + C_2 \rangle$

$-3\cos(0) + C_1 = 6$

$C_1 = 9$

$\vec{v}(t) = \langle -3\cos(t) + 9, 2e^t + 1 \rangle$

$2e^0 + C_2 = 3$

$C_2 = 1$

$\vec{s}(t) = \langle -3\sin(t) + 9t + C_3, 2e^t + t + C_4 \rangle$

$\vec{s}(t) = \langle -3\sin(t) + 9t - 9\pi, 2e^t + t - 2e^\pi \rangle$

$-3\sin(\pi) + 9\pi + C_3 = 0$

$9\pi + C_3 = 0$

$C_3 = -9\pi$

$2e^\pi + \pi + C_4 = \pi$

$-2e^\pi = C_4$

5.1 Areas and Distances

7. The speed, in m, of a rowing team during a race is collected at 3 minute intervals.

t (min)	0	3	6	9	12	15	18
v (m/min)	0	12	14	17	18	17	20

   Compute the lower and upper estimates for the distance traveled by the rowing team in the first 18 minutes of the race.

Lower: distance $= 3\text{ min}(0\text{ m/min} + 12 + 14 + 17 + 18 + 17)$

$= 3\text{ min}(78\text{ m/min})$

$= 234\text{ m}$

Upper: distance $= 3\text{ min}(12\text{ m/min} + 14 + 17 + 18 + 17 + 20)$

$= 3\text{ min}(98\text{ m/min})$

$= 294\text{ m}$

8. Estimate the area under the graph of $f(x) = x^3 + 1$ over the interval $[-1, 3]$ using four approximating rectangles and midpoints.

midpoints: $x_1 = -1/2$

$x_2 = 1/2$

$x_3 = 3/2$

$x_4 = 5/2$

$f(x_1) = (-\frac{1}{2})^3 + 1 = \frac{7}{8}$

$f(x_2) = (1/2)^3 + 1 = 9/8$

$f(x_3) = (3/2)^3 + 1 = 35/8$

$f(x_4) = (5/2)^3 + 1 = \frac{133}{8}$

$$\Delta x = \frac{3-(-1)}{4} = 1$$

$$\text{Area} \approx \Delta x \left(f(x_1) + f(x_2) + f(x_3) + f(x_4) \right)$$

$$\text{Area} \approx 1 \left(\frac{7}{8} + \frac{9}{8} + \frac{35}{8} + \frac{133}{8} \right)$$

$$\text{Area} \approx \frac{184}{8} \text{ or } 23 \text{ unit}^2$$

9. Use the definition with right endpoints to express the area under the graph of $f(x) = \dfrac{x\ln(x)}{x^2 - 3}$ over the interval $[e^2, e^5]$ as a limit.

$$\Delta x = \frac{e^5 - e^2}{n}$$

right endpoints: $x_i = e^2 + i\left(\frac{e^5 - e^2}{n}\right)$

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\Delta x \cdot f(x_i) \right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left[\left(\frac{e^5 - e^2}{n} \right) \frac{\left(e^2 + \frac{i(e^5 - e^2)}{n}\right) \ln\left(e^2 + \frac{i(e^5 - e^2)}{n}\right)}{\left(e^2 + \frac{i(e^5 - e^2)}{n}\right)^2 - 3} \right]$$

10. Determine a region whose area is equal to the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\underbrace{\frac{\pi}{4n}}_{\Delta x} \tan\left(\frac{\pi i}{4n}\right)\right)$. Do not evaluate the limit.

$\Delta x = \frac{\pi}{4n} = \frac{\pi}{4}\cdot\frac{1}{n}$ means $b-a=\frac{\pi}{4}$

$x_i = \frac{\pi i}{4n} = 0 + i\left(\frac{\pi}{4}\cdot\frac{1}{n}\right)$ means $a=0$

thus $b=\frac{\pi}{4}$

$f(x) = \tan(x)$

$f(x) = \tan(x)$ on $\left[0, \frac{\pi}{4}\right]$

5.2 The Definite Integral

11. Express the integral, $\int_2^8 \left(x\sqrt{x^2+4}\right) dx$ as a limit of Riemann sums using left endpoints. Do not evaluate the limit.

$\Delta x = \frac{8-2}{n} = \frac{6}{n}$

left end points $x_i = 2 + (i-1)\cdot\frac{6}{n}$

$$\int_2^8 \left(x\sqrt{x^2+4}\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2+(i-1)\tfrac{6}{n}\right)\sqrt{\left(2+(i-1)\tfrac{6}{n}\right)^2+4}\left(\frac{6}{n}\right)\right]$$

12. Use geometry to evaluate $\int_0^3 \left(\sqrt{9-x^2}+x\right) dx$.

Note: $\int_0^3 \left(\sqrt{9-x^2}+x\right)dx = \underbrace{\int_0^3 \sqrt{9-x^2}\,dx}_{\text{quarter of a circle with radius 3 centered at (0,0)}} + \underbrace{\int_0^3 x\,dx}_{\text{triangle with base 3 and height 3}}$

$$\int_0^3 \left(\sqrt{9-x^2}+x\right)dx \approx \frac{1}{4}\left(\pi(3)^2\right) + \frac{1}{2}(3)(3)$$

$$\approx \frac{9}{4}\pi + \frac{9}{2}$$

13. If $\int_A^B f(x)\,dx = 12$, $\int_B^A h(x)\,dx = 15$ and $\int_A^B [2f(x) - 3g(x) + 5h(x)]dx = 150$, find $\int_A^B g(x)\,dx$.

$$\int_A^B [2f(x) - 3g(x) + 5h(x)]\,dx = \int_A^B 2f(x)\,dx + \int_A^B -3g(x)\,dx + \int_A^B 5h(x)\,dx$$

$$150 = 2\int_A^B f(x)\,dx - 3\int_A^B g(x)\,dx - 5\int_B^A h(x)\,dx$$

$$150 = 2(12) - 3\int_A^B g(x)\,dx - 5(15)$$

$$201 = -3\int_A^B g(x)\,dx$$

$$-67 = \int_A^B g(x)\,dx$$

Review for Exam 3

14. Approximate $\frac{1}{\sqrt{4.1}}$.

$f(x) = \frac{1}{\sqrt{x}}$ or $x^{-1/2}$

$a = 4$

$f(a) = f(4) = \frac{1}{\sqrt{4}} = \frac{1}{2}$

$f'(x) = -\frac{1}{2}x^{-3/2}$ or $\frac{-1}{2\sqrt{x^3}}$

$f'(a) = f'(4) = \frac{-1}{2(\sqrt{4})^3} = -\frac{1}{16}$

$L(x) = f'(a)(x-a) + f(a)$

$$\frac{1}{\sqrt{4.1}} \approx -\frac{1}{16}(4.1 - 4) + \frac{1}{2}$$

$$\approx -\frac{1}{16}\cdot\left(\frac{1}{10}\right) + \frac{1}{2}$$

$$\approx \frac{-1}{160} + \frac{1}{2}$$

$$\frac{1}{\sqrt{4.1}} \approx \frac{79}{160}$$

15. Determine the differential of the function $f(x) = x^2 \ln(3x + 4)$.

$y = x^2 \ln(3x+4)$

$$dy = \left(2x\ln(3x+4) + \frac{3x^2}{3x+4}\right)dx$$

16. Determine the absolute extrema for $g(x) = (x^3 - 12x)^{\frac{1}{3}}$ on the interval $[0, 4]$.

g(x) is continuous over $[0,4]$ as g(x) is the odd root of a polynomial.

$$g'(x) = \frac{1}{3}(x^3-12x)^{-2/3} \cdot (3x^2-12) = \frac{x^2-4}{(x^3-12x)^{2/3}}$$

Critical Numbers:

$x^2-4=0$ or $(x^3-12x)^{\frac{2}{3}}=0$

$(x-2)(x+2)=0$ $\quad$ $x^3-12x=0$

$x=2$ or -2 $\quad$ $x(x^2-12)=0$

$x=0$ or $x=+\sqrt{12}$ or $x=-\sqrt{12}$

only $x=2$ and $x=\sqrt{12}$ in domain

x	$g(x)$
0	$(0^3-12(0))^{1/3}=0$
2	$(2^3-12(2))^{1/3}=\sqrt[3]{-16}$
$\sqrt{12}$	$((\sqrt{12})^3-12\sqrt{12})^{1/3}=0$
4	$(4^3-12(4))^{1/3}=\sqrt[3]{16}$

- Absolute Minimum of $\sqrt[3]{-16}$
- Absolute Maximum of $\sqrt[3]{16}$

17. Determine the absolute extrema for $h(x) = \begin{cases} 3x+4 & \text{if } -4 \le x < -1 \\ x^2 & \text{if } -1 \le x \le 0 \\ e^x - 1 & \text{if } 0 < x \le 2 \end{cases}$

$3(-1)+4=1$
$(-1)^2=1$
$(0)^2=0$
$e^0-1=0$

h(x) is continuous on $[-4,2]$

$$h'(x) = \begin{cases} 3 & \text{if } -4 < x < -1 \\ 2x & \text{if } -1 < x < 0 \\ e^x & \text{if } 0 < x < 2 \end{cases}$$

Critical Numbers:

h'(x) DNE:

$x=-1$

$x=0$

x	$h(x)$
-4	$3(-4)+4=-8$
-1	$(-1)^2=1$
0	$(0)^2=0$
2	$e^2-1=e^2-1$

- Absolute Minimum of -8
- Absolute Maximum of e^2-1

18. Determine any real number(s) c that satisfy the conclusion of the Mean Value Theorem for $f(x) = 2x - 5x^2$ on [-1,3].

$f(x)$ is a polynomial and thus continuous on the closed interval $[-1, 3]$.

$\frac{df(x)}{dx} = 2 - 10x$ is a linear function and so $f(x)$ is differentiable on the open interval $(-1, 3)$

so by the Mean Value Theorem there exists a real number c in $(-1, 3)$ such that $f'(c) = \frac{f(3) - f(-1)}{3 - (-1)}$.

$$2 - 10c = \frac{[2(3) - 5(3)^2] - [2(-1) - 5(-1)^2]}{3 - (-1)}$$

$$2 - 10c = \frac{-39 - (-7)}{4}$$

$$2 - 10c = -8$$

$$-10c = -10$$

$$\boxed{c = 1}$$

19. Given $f(x) = x^2 \ln(x)$, determine the intervals on which $f(x)$ is increasing or decreasing.

$f(x) = x^2 \cdot \ln(x)$ Note: $x > 0$

$f'(x) = 2x \ln(x) + x^2 \cdot \frac{1}{x} = x(2\ln(x) + 1)$ Note: $x > 0$

Critical Numbers:

$x(2\ln(x) + 1) = 0$

$x = 0$ or $2\ln(x) + 1 = 0$

Not in domain

$\ln(x) = -\frac{1}{2}$

$x = e^{-1/2} > 0$

behavior of $f(x)$: $\searrow$ $\nearrow$

sign of $f'(x)$: $-$ $+$

0 $e^{-1/2}$

- $f(x)$ is increasing on $(e^{-1/2}, \infty)$
- $f(x)$ is decreasing on $(0, e^{-1/2})$

20. Given $g(x) = -x^4 + 2x^2 + 5$, determine any local extrema and any inflection points for $g(x)$.

$g(x) = -x^4 + 2x^2 + 5 \qquad x \in (-\infty, \infty)$

$g'(x) = -4x^3 + 4x \qquad x \in (-\infty, \infty)$

$g''(x) = -12x^2 + 4 \qquad x \in (-\infty, \infty)$

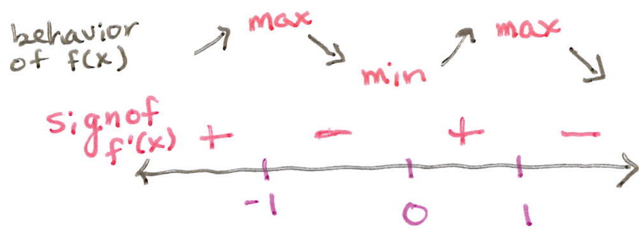
Critical Numbers for $g'(x)$

$-4x^3 + 4x = 0$

$-4x(x^2 - 1) = 0$

$x = 0$ or $x^2 - 1 = 0$

$x = -1$ or $x = 1$



• Local maximum of $g(-1) = 8$ and $g(1) = 8$

• Local minimum of $g(0) = 5$

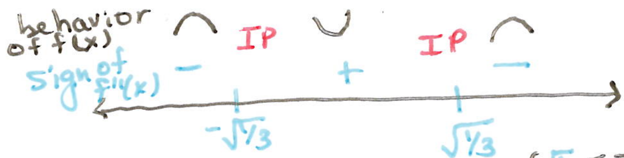
Numbers for $g''(x)$

$-12x^2 + 4 = 0$

$-12x^2 = -4$

$x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}}$



• Inflection Points at $(-\sqrt{\frac{1}{3}}, 5.78)$ and $(\sqrt{1/3}, 5.78)$

21. Compute the limit, $\lim\limits_{x\to\infty}\left(\frac{3x^3}{3x^2-4} - \frac{x^2}{x+1}\right)$ if possible.

$$\lim_{x\to\infty}\left(\frac{\overbrace{3x^3}^{\to\infty}}{\underbrace{3x^2-4}_{\to\infty}} - \frac{\overbrace{x^2}^{\to\infty}}{\underbrace{x+1}_{\to\infty}}\right) = \lim_{x\to\infty}\left(\frac{3x^3(x+1) - x^2(3x^2-4)}{(3x^2+4)(x+1)}\right)$$

?

$$= \lim_{x\to\infty}\left(\frac{\overbrace{3x^3 + 4x^2}^{\to\infty}}{\underbrace{3x^3 + 3x^2 + 4x + 1}_{\to\infty}}\right)$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty}\left(\frac{\overbrace{9x^2 + 8x}^{\to\infty}}{\underbrace{9x^2 + 6x + 4}_{\to\infty}}\right)$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty}\left(\frac{\overbrace{18x + 8}^{\to\infty}}{\underbrace{18x + 6}_{\to\infty}}\right)$$

$$\overset{\text{L.H}}{=} \lim_{x\to\infty}\left(\frac{18}{18}\right)$$

$= 1$

22. Compute the limit, $\lim_{x\to-\infty}\left(\dfrac{e^{-x^3}}{x^2}\right)$ if possible.

$$\lim_{x\to-\infty}\left(\dfrac{\overbrace{e^{-x^3}}^{\to\infty}}{\underbrace{x^2}_{\to\infty}}\right) \overset{\text{L.H.}}{=} \lim_{x\to-\infty}\left(\dfrac{e^{-x^3}\cdot(-3x^2)}{2x}\right)$$

$$= \lim_{x\to-\infty}\left(-\dfrac{3}{2}\underbrace{x\,e^{-x^3}}_{\to\infty}\right)$$

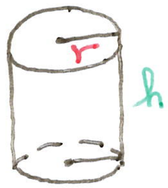
$$\to \infty$$

23. Compute the limit, $\lim_{x\to1}\left(\dfrac{5\ln(x)+4x-4}{e^{3x-3}+7x^2-8}\right)$ if possible.

$$\lim_{x\to1}\left(\dfrac{5\overset{\to0}{\ln(x)}+\overbrace{4x-4}^{\to0}}{\underbrace{e^{3x-3}+7x^2-8}_{1+7-8\,\to0}}\right) \overset{\text{L.H.}}{=} \lim_{x\to1}\left(\dfrac{\overbrace{\frac{5}{x}+4}^{\to9}}{\underbrace{3e^{3x-3}+14x}_{\to17}}\right)$$

$$= \dfrac{9}{17}$$

24. A cylindrical storage container has a volume of 108π m^3. The material for the base and top costs \$2 per square meter and the material for the side costs \$1 per square meter. Determine the minimum cost for the storage container.



$V = 108\pi \text{ m}^3$

$\pi r^2 h = 108\pi$

$h = \frac{108}{r^2}$ $\quad r$ in $(0,\infty)$

Minimize Cost:

Cost = 2 · 2 bases + 1 · 1 side

$\text{Cost} = 2\cdot 2(\pi r^2) + 1\cdot 1\cdot(2\pi r h)$

$\text{Cost} = 4\pi r^2 + 2\pi r h$

$C(r) = 4\pi r^2 + 2\pi r\left(\frac{108}{r^2}\right)$

$C(r) = 4\pi r^2 + \frac{216\pi}{r}$

$\frac{dC(r)}{dr} = 8\pi r - \frac{216\pi}{r^2}$

Critical Numbers:

$0 = 8\pi r - \frac{216\pi}{r^2}$ $\quad r \neq 0$

$0 = 8\pi r^3 - 216\pi$

$216\pi = 8\pi r^3$

$27 = r^3$

$3 = r$

$\frac{d^2C(r)}{dr^2} = 8\pi + \frac{432\pi}{r^3}$

$\left.\frac{d^2C(r)}{dr^2}\right|_{r=3} = 8\pi + \frac{432\pi}{(3)^3} > 0$ so min

$C(3) = 4\pi(3)^2 + \frac{216\pi}{3}$

$= 36\pi + 72\pi$

$= 108\pi$

The minimum cost is \$$108\pi$ or $\approx$ \$339.29 when the radius is 3 m and the height is 12 m.