Math 308: Week-in-Review 13 (7.7 - 7.9)

7.7-7.9: Nonhomogeneous Linear Systems

Review

- For a nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$, the general solution is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h$ is the general solution to the homogeneous system $\mathbf{x}' = A\mathbf{x}$, and $\mathbf{x}_p$ is a particular solution to the nonhomogeneous system.
- **Variation of Parameters:**
  - This method is general and can be used for any continuous $\mathbf{g}(t)$.
  - Assume $\mathbf{x}_p(t) = \mathbf{\Psi}(t)\mathbf{c}(t)$, where $\mathbf{\Psi}(t)$ is a fundamental matrix for the homogeneous system.
  - Then, $\mathbf{c}'(t) = \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)$, so $\mathbf{c}(t) = \int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt$.
  - Thus, $\mathbf{x}_p(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt$.
  - This method can be computationally intensive.
- **Undetermined Coefficients:**
  - This method is applicable when $\mathbf{g}(t)$ is a vector of polynomials, exponentials, sines, cosines, or linear combinations or products thereof.
  - Guess the form of $\mathbf{x}_p(t)$ based on the form of $\mathbf{g}(t)$:
    * For $\mathbf{g}(t) = \mathbf{a}e^{kt}$, guess $\mathbf{x}_p(t) = \mathbf{b}e^{kt}$.
    * For $\mathbf{g}(t) = \mathbf{a}t^m$, guess $\mathbf{x}_p(t) = \mathbf{b}_m t^m + \mathbf{b}_{m-1}t^{m-1} + \cdots + \mathbf{b}_0$.
    * For $\mathbf{g}(t) = \mathbf{a}\sin(\omega t) + \mathbf{c}\cos(\omega t)$, guess $\mathbf{x}_p(t) = \mathbf{b}\sin(\omega t) + \mathbf{d}\cos(\omega t)$.
  - If the guessed form is already part of the homogeneous solution, multiply the guess by t (or higher powers if necessary) to obtain a linearly independent form.
  - This method is often simpler than variation of parameters when applicable, but requires careful selection of the form of $\mathbf{x}_p(t)$.

1. Suppose

$$\mathbf{x}_1 = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

are two indepedent solutions to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$, and let

$$\Psi(t) = \begin{pmatrix} \varphi_1(t) & \psi_1(t) \\ \varphi_2(t) & \psi_2(t) \end{pmatrix}$$

be a fundamental matrix. Show that $\Psi'(t) = A\Psi(t)$. Show that the general solution of the homogeneous system can be written equivalently as

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \Psi(t)\mathbf{c}$$

where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is an arbitrary constant vector.

2. Consider the nonhomogeneous equation

$$\mathbf{x}' = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$

Find the fundamental matrix and its inverse. Find a particular solution to the system and the general solution.

3. Consider the non-homogeneous system

$$\mathbf{x}' = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} te^t \\ e^t \end{bmatrix}, \quad t > 0.$$

Verify that $\Psi(t) = \begin{bmatrix} e^t & \frac{t^2}{2}e^t \\ 0 & e^t \end{bmatrix}$ is a fundamental matrix. Then find the general solution of the system.

4. Consider the system

$$x' = 3x + 2y + 3, \quad y' = 7x + 5y + 2t$$

(a) Find the fundamental matrix (b) Use undetermined coefficients to find a particular solution.

5. Use the Method of Undetermined Coefficients to find the general solution solution of the system

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 + e^t \\ -2 - 2e^t \end{bmatrix}$$

6. Use the Method of Undetermined Coefficients to determine the general solution of

$$\mathbf{x}' = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} e^t$$

7. **Without solving for the coefficients**, determine the form of the particular solution $\mathbf{x}_p$ of the nonhomogeneous system

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2\cos t + 4\sin t \\ 2\sin t \end{bmatrix}$$