MATH 308: WEEK-IN-REVIEW 2 (2.2& 2.3)

Section 2.2 (Separation of variables) $\frac{dy}{dx} = g(x)h(y)$ * separable *

1. Use separation of variables to find the general solution to the following differential equations.

* separable

(a) $\frac{dy}{dx} = 3y + 2 = 3(y + 2/3)$

Equilibrium soln

$3y + 2 = 0 \Rightarrow y = -2/3$

$\boxed{y(x) = -2/3}$

$$\int \frac{1}{y + 2/3}\,dy = \int 3\,dx$$

$$\Rightarrow \ln|y + 2/3| = 3x + C$$

$$\Rightarrow |y + 2/3| = e^{C} \cdot e^{3x}$$

$$\Rightarrow y + 2/3 = \pm e^{C} \cdot e^{3x} = K e^{3x}$$

$(K = \pm e^{C})$ $\Downarrow$ $K \neq 0$

$$y = -2/3 + K e^{3x}$$

$$\boxed{y(x) = -2/3 + K e^{3x}}$$

where K is any real number

* the solution $y(x) = -2/3$ corresponds to $K = 0$ *

(b) $\frac{dg}{dx} = \frac{2g}{x+1}$ * separable

Equilibrium soln

$2g = 0 \Rightarrow \boxed{g(x) = 0,\ x \neq -1}$

$$\frac{1}{g}\,dg = 2\,\frac{1}{x+1}\,dx$$

$$\Rightarrow \int \frac{1}{g}\,dg = 2\int \frac{1}{x+1}\,dx$$

$$\Rightarrow \ln|g| = 2\ln|x+1| + C = \ln(x+1)^2 + C$$

$$\Rightarrow |g| = e^{C + \ln(x+1)^2} = e^{C} \cdot e^{\ln(x+1)^2}$$

$$g(x) = \pm e^{C} \cdot (x+1)^2 = K(x+1)^2$$

general solution

$g(x) = K(x+1)^2$ where K is any real number

* the equilibrium soln $g(x) = 0$ corresponds to $K = 0$ *

(c) $\dfrac{d\theta}{dt} = t\sqrt{t^2+1}\sec(\theta), \qquad \theta(0) = \dfrac{\pi}{6}$

Equilibrium solns

$\sec(\theta)=0 \Rightarrow \dfrac{1}{\cos(\theta)} = 0$ (no solutions)

$\displaystyle\int \frac{1}{\sec(\theta)}\,d\theta = \int t\sqrt{t^2+1}\,dt$ * separable

$\Rightarrow \displaystyle\int \cos(\theta)\,d\theta = \int t\sqrt{t^2+1}\,dt$ * $u = t^2+1$

$\Rightarrow \sin(\theta) = \dfrac{1}{2}\displaystyle\int 2t\sqrt{t^2+1}\,dt = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}u^{3/2}\cdot \frac{2}{3} + C$

$= \dfrac{1}{3}(t^2+1)^{3/2} + C$

$\Rightarrow \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{3}(0^2+1)^{3/2} + C \Rightarrow \dfrac{1}{2} = \dfrac{1}{3} + C \Rightarrow C = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$

$\Rightarrow \boxed{\sin(\theta) = \dfrac{1}{3}(t^2+1)^{3/2} + \dfrac{1}{6}}$ or $\boxed{\theta(t) = \arcsin\left[\dfrac{1}{3}(t^2+1)^{3/2} + \dfrac{1}{6}\right]}$

implicit   explicit

(d) $\dfrac{dy}{dt} = \dfrac{2t}{y+yt^2}, \qquad y(2) = 3.$ Determine the values of t where the solution is defined.

$= \dfrac{2t}{y(1+t^2)}$ $u = 1+t^2$

$\Rightarrow \displaystyle\int y\,dy = \int \frac{2t}{1+t^2} = \int \frac{1}{u}\,du = \ln|u| + C$

$\Rightarrow \dfrac{y^2}{2} = \ln(1+t^2) + C \Rightarrow y^2 = 2\ln(1+t^2) + 2C$

$= 2\ln(1+t^2) + K$

$3^2 = 2\ln(5) + K \Rightarrow K = 9 - 2\ln(5)$

$y^2 = 2\ln(1+t^2) + 9 - 2\ln(5)$

$\boxed{y = \sqrt{2\ln(1+t^2) + 9 - 2\ln(5)}}$

* separable *

2. Solve the differential equation $\dfrac{dy}{dt} = (y^2 - 9)\cos(t)$, $y(0) = 6$. Leave your solution in both implicit and explicit forms.

Equilibrium solns

$$y^2 - 9 = 0 \Rightarrow y(t) = \pm 3$$

$$\int \frac{1}{y^2-9}\,dy = \int \cos(t)\,dt$$

$$\Rightarrow \int \frac{1}{(y-3)(y+3)}\,dy = \int \cos(t)\,dt = \sin(t) + C$$

* partial fractions *

$$\frac{1}{(y-3)(y+3)} = \frac{A}{y-3} + \frac{B}{y+3} \Rightarrow 1 = A(y+3) + B(y-3)$$

$$\Rightarrow \begin{cases} * \ y = 3: \ 1 = A(6) + B(0) \Rightarrow A = 1/6 \\ * \ y = -3: \ 1 = A(0) + B(-6) \Rightarrow B = -1/6 \end{cases}$$

$$\int \frac{1}{(y-3)(y+3)}\,dy = \frac{1}{6}\int \frac{1}{y-3}\,dy - \frac{1}{6}\int \frac{1}{y+3}\,dy = \frac{1}{6}\ln|y-3| - \frac{1}{6}\ln|y-3|$$
$$= \sin(t) + C$$

$$\frac{1}{6}\left(\ln|y-3| - \ln|y+3|\right) = \sin(t) + C \Rightarrow \frac{1}{6}\ln\left|\frac{y-3}{y+3}\right| = \sin(t) + C$$

$$\Rightarrow \ln\left|\frac{y-3}{y+3}\right| = 6\sin(t) + 6C$$

$$\Rightarrow \left|\frac{y-3}{y+3}\right| = e^{6\sin(t) + 6C} = e^{6C} \cdot e^{6\sin(t)}$$

$$K = \pm e^{6C}$$

$$\Rightarrow \frac{y-3}{y+3} = \pm e^{6C} \cdot e^{6\sin(t)} = K e^{6\sin(t)}$$

$\Rightarrow$ Find K: $\quad \dfrac{6-3}{6+3} = K e^{\sin(0)} \Rightarrow K = \dfrac{1}{3} \Rightarrow$ $\boxed{\dfrac{y-3}{y+3} = \dfrac{1}{3} e^{6\sin(t)}}$ implicit

$$\boxed{y = \frac{3 + e^{6\sin(t)}}{1 - \frac{1}{3}e^{6\sin(t)}}}$$ explicit

3. (a) Find the general solution to the differential equation $\dfrac{dy}{dt} = 2ty^2$. * separable *

   (b) Find the specific solution that satisfies the initial condition $y(0) = -1$. Determine the interval where the solution is defined.

   (c) Find the specific solution that satisfies the initial condition $y(0) = 1$. Determine the interval where the solution is defined.

(a) $\dfrac{dy}{dt} = 2ty^2 \Rightarrow \int \frac{1}{y^2}dy = 2\int t\,dt = t^2 + C$

$\Rightarrow -\dfrac{1}{y} = t^2 + C \Rightarrow \dfrac{1}{y} = -(t^2+C) \Rightarrow \boxed{y = -\dfrac{1}{t^2+C}}$

(b) If $y(0) = -1$, $t = 0$ & $y = -1$

$-1 = -\dfrac{1}{0^2+C} \Rightarrow -1 = -\dfrac{1}{C} \Rightarrow C = 1 \Rightarrow \boxed{y(t) = -\dfrac{1}{t^2+1}}$

$\boxed{\text{Domain: } t \in (-\infty, \infty) \Rightarrow t \text{ is any real \#}}$

(c) If $y(0) = 1$, $t = 0$ & $y = 1$ $\Rightarrow y(t) = -\dfrac{1}{t^2-1}$

$1 = -\dfrac{1}{0^2+C} \Rightarrow C = -1$

$\boxed{\text{Domain: } t \in (-1, 1)}$

Section 2.3 (Modeling with First-order ODEs) $\frac{dP}{dt} \propto P \Leftrightarrow \frac{dP}{dt} = kP$

4. A population of bacteria grows (k>0) at a rate proportional to its current size. Initially, there are 100 bacteria, and after 2 hours, the population has grown to 500 bacteria.

   (a) Write the differential equation that models the population growth.
   (b) Solve the differential equation to find the population as a function of time.
   (c) Determine the population after 5 hours.
   (d) How long will it take for the population to reach 10,000 bacteria?

(a) Let $P = P(t)$ be the population at time t

$$\boxed{\frac{dP}{dt} = kP, \quad P(0) = 100}$$ $k > 0$ growth constant

(b) $\frac{dP}{dt} = kP$ is separable

$C = \pm e^c$

$$\int \frac{1}{P}\,dP = k\int dt \Rightarrow \ln|P| = kt + c$$

$$\Rightarrow |P| = e^c \cdot e^{kt} \Rightarrow P = \pm e^c \cdot e^{kt} = Ce^{kt}$$

$$P(t) = Ce^{kt}, \quad P(0) = C = 100 \Rightarrow P(t) = 100e^{kt}$$

Find k: $P(2) = 500 = 100e^{2k} \Rightarrow \frac{500}{100} = e^{2k} \Rightarrow \ln(5) = 2k$

$\Rightarrow k = \frac{\ln(5)}{2}$

$$\boxed{P(t) = 100e^{\frac{t}{2}\ln(5)}} \text{ or } \boxed{P(t) = 100 \cdot 5^{t/2}}$$

(c) $P(5) = 100e^{\frac{5}{2}\ln(5)} = 100 \cdot 5^{5/2} = 5{,}590$ bacteria after 5 hrs

(d) $10{,}000 = 100 \cdot 5^{t/2} \Rightarrow 100 = 5^{t/2} \Rightarrow \ln(100) = \frac{t}{2}\ln(5)$

$$\boxed{t = \frac{2\ln(100)}{\ln(5)} = 5.72 \text{ hrs}}$$

5. In deep water, the intensity of light is given by a function, $I = I(x)$, where I is measured in units like W/m^2, and x is the depth in meters. The differential equation describing light intensity is $I' = -kI$, where k depends on properties like water clarity or the presence of particles. If $k = 0.8$ per meter and the intensity of light at the surface of a lake is 1000 W/m^2

   (a) Solve the differential equation that describes the light intensity at depth x.

   (b) Determine the light intensity at a depth of 10 meters below the water surface.

   (c) At what depth is the light intensity 250 W/m^2?

(a) $\frac{dI}{dt} = -0.8I$, $I(0) = 1000$ (exponential decay)

$$\int \frac{1}{I} dI = -0.8 \int dx \Rightarrow I = e^{c} \cdot e^{-0.8x} = Ke^{-0.8x}$$

$$\ln|I| = -0.8x + C$$

$$I(0) = K = 1000$$

$$\boxed{I(x) = 1000e^{-0.8x}}$$

(b) $I(10) = 1000e^{-0.8(10)} = 1000e^{-8} = \boxed{0.3355 \text{ W/m}^2}$

(c) $250 = 1000e^{-0.8x} \Rightarrow \frac{250}{1000} = e^{-0.8x} \Rightarrow 0.25 = e^{-0.8x}$

$$\ln(0.25) = -0.8x \Rightarrow x = -\frac{\ln(0.25)}{0.8} = \boxed{1.73 \text{ m}}$$

6. A tank initially contains 100 liters of pure water. A brine solution with a concentration of 0.5 kg/L of salt flows into the tank at a rate of 2 L/min. The well-stirred mixture flows out of the tank at the same rate. Find the amount of salt in the tank at any time t.

Let $q(t)$ = mass of salt at time t, $q(0)=0$

$$\frac{dq}{dt} = \text{rate in} - \text{rate out} = \left(0.5\,\frac{g}{L}\right)(2\,L/min) - \left(\frac{q}{100}\,\frac{g}{L}\right)(2\,L/min)$$

$$\frac{dq}{dt} = 1 - \frac{1}{50}q \Rightarrow \boxed{\frac{dq}{dt} + \frac{1}{50}q = 1,\ q(0)=0}$$

$$p(t) = \frac{1}{50},\ g(t) = 1 \Rightarrow \mu(t) = e^{t/50}$$

$$\underbrace{e^{t/50}\frac{dq}{dt} + \frac{1}{50}e^{t/50}q}_{\frac{d}{dt}\left[e^{t/50}q\right]} = e^{t/50}$$

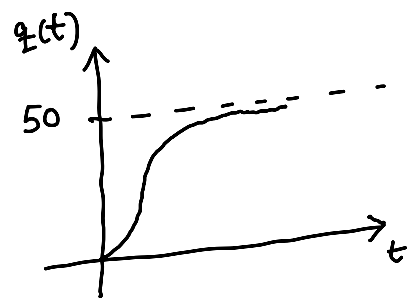
$$\frac{d}{dt}\left[e^{t/50}q\right] = e^{t/50} \Rightarrow e^{t/50}q = \int e^{t/50}\,dt = 50e^{t/50} + C$$

$$q(t) = \frac{50e^{t/50} + C}{e^{t/50}} = 50 + Ce^{-t/50}$$

$$q(0) = 0 = 50 + C \Rightarrow C = -50$$

$$q(t) = 50 - 50e^{-t/50}$$

$$\boxed{q(t) = 50\left(1 - e^{-t/50}\right)}$$



7. A tank initially contains 200 liters of water with 10 kg of salt dissolved in it. A brine solution with a concentration of 0.2 kg/L flows into the tank at a rate of 3 L/min. The well-stirred mixture flows out at a rate of 2 L/min. Find the amount of salt in the tank at any time t.

$q(0) = 10\,kg$, $V(0) = 200\,L$ * volume *

$$V(t) = 200 + (3-2)t = 200 + t$$

$$\frac{dq}{dt} = (0.2)(3) - \frac{q \cdot 2}{200+t} = 0.6 - \frac{2q}{200+t}$$

$$\boxed{\frac{dq}{dt} + \frac{2q}{200+t} = 0.6, \quad q(0) = 10}$$

$$p(t) = \frac{2}{200+t}, \quad g(t) = 0.6, \quad \mu(t) = e^{\int \frac{2}{200+t}dt} = e^{2\ln(200+t)} = e^{\ln(200+t)^2} = (200+t)^2$$

$$\underbrace{(200+t)^2 \frac{dq}{dt} + 2(200+t)q}_{} = 0.6(200+t)^2$$

$$\frac{d}{dt}\left[(200+t)^2 q\right] = 0.6(200+t)^2$$

$$\Rightarrow (200+t)^2 q = 0.6\int (200+t)^2 dt = \frac{0.6}{3}(200+t)^3 + C$$

$$\Rightarrow q(t) = 0.2(200+t) + \frac{C}{(200+t)^2}$$

$$\Rightarrow q(0) = 0.2(200) + \frac{C}{200^2} = 10 \Rightarrow 40 - 10 = -\frac{C}{200^2}$$

$$\Rightarrow C = -30 * 200^2$$

$$\boxed{q(t) = 0.2(200+t) - \frac{30 * 200^2}{(200+t)^2}}$$

8. A car engine is turned off after running at 90°. The ambient temperature is 15°. After 15 minutes, the engine cools to 60°. What is the temperature of the engine after 45 minutes?

Newton's Law of Cooling

$u(t)$ = temperature of engine at time t

$\frac{du}{dt} = -k(u-T)$, $u(0) = 90^\circ C$

T = ambient temperature $= 15^\circ C$

$\frac{du}{dt} = -k(u-15)$

$\int \frac{1}{u-15}\,du = -k\int dt$

$\ln|u-15| = -kt + C$

$|u-15| = e^{C}\cdot e^{-kt} \Rightarrow u-15 = \pm e^{C}\cdot e^{-kt}$

$u-15 = Ce^{-kt} \Rightarrow u = 15 + Ce^{-kt}$

$u(0) = 90 = 15 + C \Rightarrow C = 75 \Leftrightarrow \boxed{u = 15 + 75e^{-kt}}$

* Find k * $u(15) = 60 = 15 + 75e^{-15k}$

$45 = 75e^{-15k} \Leftrightarrow \frac{45}{75} = e^{-15k}$

$\frac{3}{5} = e^{-15k} \Rightarrow \ln(3/5) = -15k \Rightarrow k = -\frac{1}{15}\ln(3/5)$

$u(t) = 15 + 75e^{-(-\frac{1}{15}\ln(3/5))t}$

$= 15 + 75e^{t/15\ln(3/5)}$

$u(t) = 15 + 75\left(\frac{3}{5}\right)^{t/15}$

$u(45) = 15 + 75\left(\frac{3}{5}\right)^{\frac{45}{15}}$

$= 15 + 75\left(\frac{3}{5}\right)^{3} \approx \boxed{31.2\,^\circ C}$