Math 308: Week-in-Review 12 (7.5 - 7.6)

7.5: Homogeneous Linear Systems with Constant Coefficients

Review

- How to solve a homogeneous linear system with constant coefficients (when you have distinct real eigenvalues)
  1. Assume your solution has the form $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$.
  2. Plug this in to get an eigenvalue problem.
  3. Solve for the eigenvalues r_1 and r_2 and the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
  4. The general solution is $c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$.
- Phase plane/portrait: A phase plane/portrait is essentially a 2D version of the phase line. It shows you where the solution moves as time passes.
- An equilibrium point is a point where if you start there, you will remain there forever. The origin is always an equilibrium point of the differential equation system $\mathbf{x}' = A\mathbf{x}$.
- Stability of equilibrium points
  - Asymptotically stable: If you start near the equilibrium point, you will be sucked into it as $t \to \infty$.
  - Stable: If you start near the equilibrium point, you will stay near it.
  - Unstable: There is at least one point near the equilibrium point that goes away from the equilibrium point.

7.6: Complex Eigenvalues

Review

- To solve the system $\mathbf{x}' = A\mathbf{x}$ when you have complex eigenvectors:
  - Solve for just one of the eigenvectors.
  - Separate $\boldsymbol{\xi} e^{rt}$ into its real and imaginary parts.
  - The real and imaginary parts form a fundamental set of solutions.
  - (Assuming that A is 2×2. If A is larger, then there are also more solutions.)

1. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \overset{A}{\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}} \mathbf{x} \qquad X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

$$\operatorname{tr} A = 1 + 4 = 5, \quad \det A = 1.4 - (-1 \cdot 2) = 4 + 2 = 6$$

$$\lambda^2 - 5\lambda + 6 = 0, \quad (\lambda - 3)(\lambda - 2) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = 2$$

Eigenvectors: $\lambda_1 = 3$: $(A - 3I)V_1 = \vec{0}$

$$\begin{pmatrix} 1-3 & 2 \\ -1 & 4-3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-a + b = 0 \Rightarrow a = b$$

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2: \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -a + 2b = 0 \Rightarrow a = 2b$$

$$V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$$

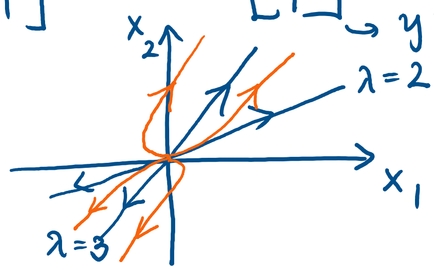
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

as $t \to \infty$

$e^{3t} > e^{2t}$

$y = \frac{1}{2}x$

$y = x$



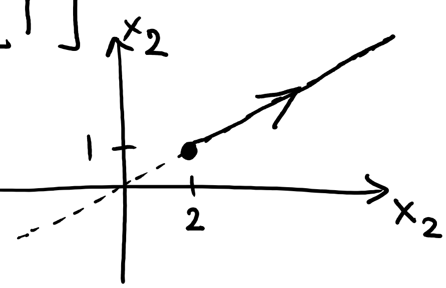
unstable node

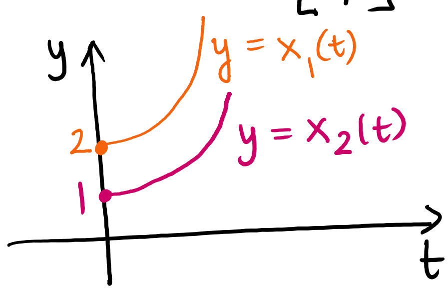
2. Solve the initial value problem when $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Draw this solution on the phase plane and sketch the graph of $x_1(t)$ and $x_2(t)$.

$$x(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow c_1 = 0, \ c_2 = 1$$

$$x(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$$





3. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x_1' = -5x_1 + 4x_2$$
$$x_2' = \frac{3}{2}x_1 - 4x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -5 & 4 \\ 3/2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\text{tr}\, A = -9$

$\det A = 14$

$$\lambda^2 - (\text{tr}A)\lambda + \det A = 0 \Rightarrow \lambda^2 + 9\lambda + 14 = 0$$

$$(\lambda+7)(\lambda+2) = 0 \Rightarrow \lambda_1 = -7, \ \lambda_2 = -2$$

$\lambda_1 = -7$: $\begin{pmatrix} 2 & 4 \\ 3/2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a + 4b = 0$

$a = -2b$

$V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda_1 = -2$: $\begin{pmatrix} -3 & 4 \\ 3/2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3a + 4b = 0$

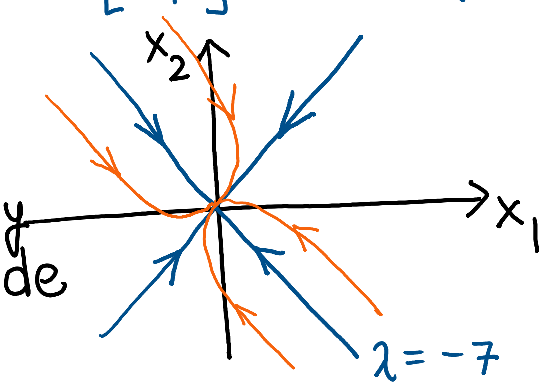
$3a = 4b$

$V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$X(t) = c_1 e^{-7t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$e^{-7t} > e^{-2t}$

$t \to -\infty$



asymptotically stable node

4. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x' = x + 2y$$
$$y' = -2x + 5y$$

$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\operatorname{tr} A = 6$

$\det A = 9$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$ (repeated)

$\lambda = 3$: $(A - 3I)v = 0 \Rightarrow \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$a = b$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ only one eigenvector

$\rightarrow$ generalized eigenvector

$v_2 = t v_1 + u$, where $(A - 3I)u = v_1$

$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow -2u_1 + 2u_2 = 1$

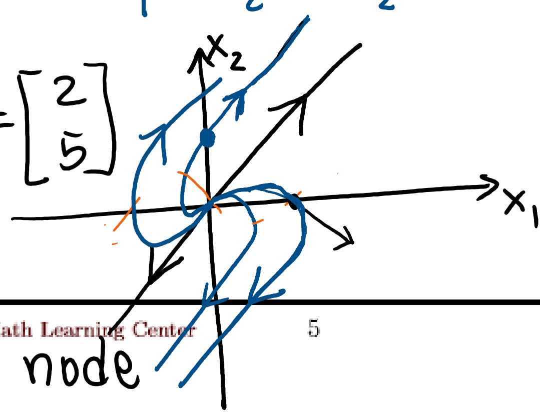
$2u_2 = 1 + 2u_1$

$u_1 = 0 \Rightarrow u_2 = 1/2 \Rightarrow u = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

$v_2 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

$x(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2 = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right]$

$\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$



$\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

unstable

deficient node

5. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x' = 3x + y$$
$$y' = -2x + y$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\text{tr}A = 4$

$\det A = 5$

$$\lambda^2 - 4\lambda + 5 = 0, \quad \lambda = \frac{4 \pm \sqrt{4^2 - 20}}{2} = 2 \pm i$$

$$\lambda = 2 + i: \begin{pmatrix} 3-(2+i) & 1 \\ -2 & 1-(2+i) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} (1-i)a + b = 0 \\ b = (i-1)a \end{matrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ i-1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{a} + i\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b}$$

$$\begin{cases} \lambda = m + in \\ V = a + ib \end{cases}$$

$$X(t) = c_1 e^{mt}\left[a\cos(nt) - b\sin(nt)\right] + c_2 e^{mt}\left[b\cos(nt) + a\sin(nt)\right]$$

$$X(t) = c_1 e^{2t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)\right]$$

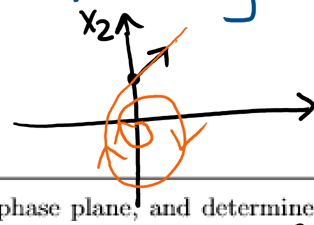
$$+ c_2 e^{2t}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(t)\right]$$

$e^{2t}\left[\cos(t) \ldots\right] \quad e^{2t}\left[\sin(t) \ldots\right]$

$$+ c_2 e^{2t} \left[\begin{pmatrix} 1 \end{pmatrix} \cdots \right]$$

$$= c_1 e^{2t} \begin{bmatrix} \cos(t) \\ -\cos(t) - \sin(t) \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin(t) \\ \cos(t) - \sin(t) \end{bmatrix}$$

unstable spiral.



$$\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

6. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$$\lambda = 2i: \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2ia - 2b = 0$$

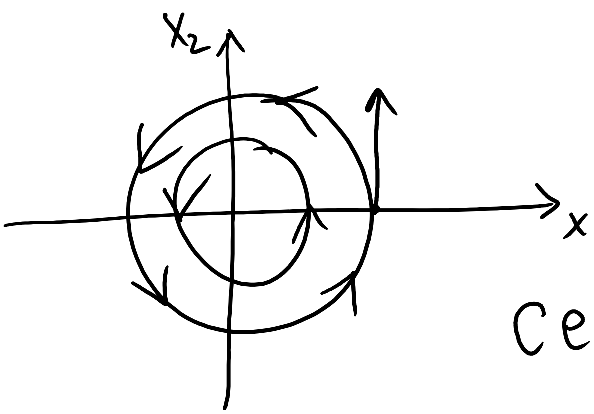
$$2a - 2ib = 0$$

$$V = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \cancel{2}a = \cancel{2}ib$$

$$X(t) = c_1 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) \right]$$

$$+ c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) \right]$$

$$= \begin{bmatrix} -c_1 \sin(2t) + c_2 \cos(2t) \\ c_1 \cos(2t) + c_2 \sin(2t) \end{bmatrix}$$



$$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Center

7. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin. Solve the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}}_{A}\mathbf{x}$$

$\lambda^2 - (\text{tr} A)\lambda + \det A = 0$ $\quad$ $\text{tr} A = -2$, $\det A = 5$

$\lambda^2 + 2\lambda + 5 = 0$

$\Rightarrow (\lambda+1)^2 + 4 = 0 \Rightarrow (\lambda+1)^2 = -4 \Rightarrow \lambda + 1 = \pm 2i \Rightarrow \boxed{\lambda = -1 \pm 2i}$

stable spiral

$$\begin{pmatrix} 1-(-1+2i) & -8 \\ 1 & -3-(-1+2i) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2-2i & -8 \\ 1 & -2-2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a = 2(1+i)b \Rightarrow v_1 = \begin{pmatrix} 2(1+i) \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

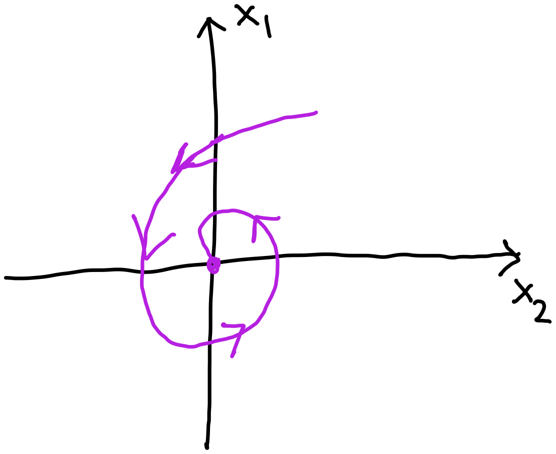
$$x(t) = c_1 e^{-t}\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos(2t) - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin(2t)\right] + c_2 e^{-t}\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos(2t) + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin(2t)\right]$$

$\begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \end{pmatrix}$ direction vector @ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x(0) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$c_1 = 2$, $4 + 2c_2 = -1$

$2c_2 = -5 \Rightarrow c_2 = -5/2$

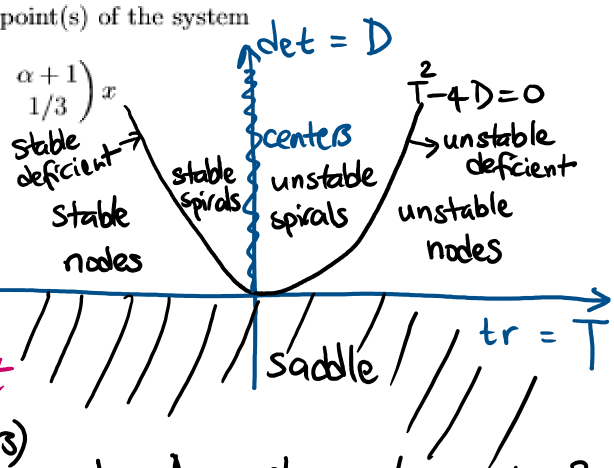


$$x(t) = 2e^{-t}\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos(2t) - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin(2t)\right] - \frac{5}{2}e^{-t}\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos(2t) + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin(2t)\right]$$

8. Classify the types and stability of the equilibrium point(s) of the system

$$x' = \begin{pmatrix} \alpha - 1 & \alpha + 1 \\ -2/3 & 1/3 \end{pmatrix} x$$

for different values of the parameter α.



$$\lambda^2 - (\operatorname{tr} A)\lambda + \det(A) = 0$$

$$\lambda = \frac{1}{2}\left[\operatorname{tr} A \pm \sqrt{\underbrace{(\operatorname{tr} A)^2 - 4\det A}_{\text{discriminant}}}\right]$$

* $T^2 - 4D < 0$ (spirals, centers)
* $T^2 - 4D \geq 0$ (nodes)
* $\det A < 0$ (saddle)

$$\operatorname{tr} A = \alpha - 1 + \frac{1}{3} = \alpha - \frac{2}{3}$$

$$\det A = \frac{1}{3}(\alpha - 1) + \frac{2}{3}(\alpha + 1)$$
$$= \alpha + \frac{1}{3}$$

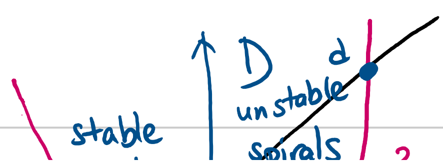
saddle: $\det A < 0$

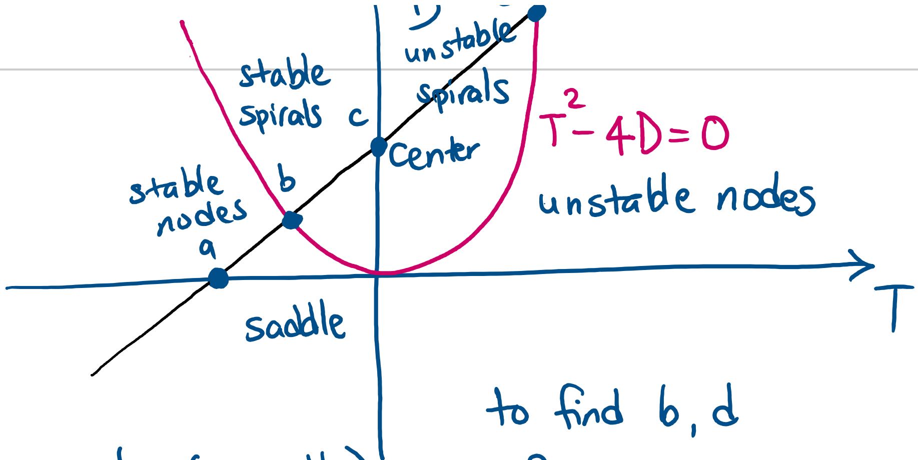
$$\alpha + \frac{1}{3} < 0$$

$$\alpha < -\frac{1}{3}$$

$$(\operatorname{tr} A, \det A) = (T, D) = \left(\alpha - \frac{2}{3}, \alpha + \frac{1}{3}\right)$$

$$D = T + 1$$





to find b, d

$$T^2 - 4D = 0$$

$$T^2 - 4(T+1) = 0$$

$$T^2 - 4T - 4 = 0$$

$$T = \frac{4 \pm \sqrt{16+16}}{2}$$

$$= 2 \pm \sqrt{8}$$

(a) $\alpha < -\frac{1}{3}$ (saddle)

(b) $T = 2 - \sqrt{8} = \alpha - 2/3$

(d) $T = 2 + \sqrt{8} = \alpha - 2/3$

(b) $\alpha = \frac{8}{3} - \sqrt{8}$

(d) $\alpha = \frac{8}{3} + \sqrt{8}$

(c) $T = 0 \Rightarrow \alpha - 2/3 = 0 \Rightarrow \alpha = 2/3$