WIR#2

Section 1.3

- **Limits at Infinity:** We write $\lim_{x \to \pm\infty} f(x) = L$ if $f(x)$ approaches the number L as x increases (or decreases) without bound. We refer to the line $y = L$ as a **horizontal asymptote**.

- **Limits at Infinity of Polynomial Functions:** To determine the end behavior of a polynomial, we must look at the leading term (i.e. the term with the highest power of x). If, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, where n is a postive integer, then $\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n = \pm\infty$ and $\lim_{x \to -\infty} p(x) = \lim_{x \to -\infty} a_n x^n = \pm\infty$

  Note: We could also remember the information from the table in the Section 1.3 Notes.

- **Limits at Infinity of Rational Functions:** If $f(x)$ is a rational function, to determine the limits at infinity, we divide every term in the function by the highest power of x in the denominator, simplify each term, and then observe the behavior of each term.

  Note: Keep in mind that if the limit at ∞ or at $-\infty$ is finite for a rational function, the limit at the other end is the same finite number. This means that if the graph of a rational function has a horizontal asymptote at one end, the graph will have the same horizontal asymptote at the other end. Please keep in mind that this is true for rational functions but not true in general.

- **Limits at Infinity of Functions Involving Exponentials:** To evaluate limits at infinity of functions involving exponentials you must first be familiar with the graphs of e^x and e^{-x}. This process is slightly different depending on whether you are looking at the limit as $x \to \infty$ or $x \to -\infty$:
  - If $x \to \infty$, divide every term in the function by the most positive power of e^{nx} in the denominator.
  - If $x \to -\infty$, divide every term in the function by the most negative power of e^{nx} in the denominator.
  - Simplify the function and then observe the behavior of each term of the function.
  - Note: If there is no positive exponent of e^{nx} in the denominator when $x \to \infty$ or no negative exponent of e^{nx} in the denominator when $x \to -\infty$, we do not divide by anything. We instead just observe the behavior of each term of the function.

- **Determining Vertical Asymptotes and Holes of a Rational Function:** Determine the value(s) of x that are not in the domain of the function (i.e. set the denominator equal to zero). We know that the graph of $f(x)$ will either have a hole or a vertical asymptote at each of those value(s) of x. We have two methods to determine what is occurring at each value of x:
  - **Option 1:** Evaluate the limit of the function at those particular value(s) of x:
    - If the limit of the function exists (i.e. is a finite number) for that particular value of x, then there is a hole at that value of x.
    - If the limit of the function does not exist for that particular value of x, then there is a vertical asymptote at that value of x.
  - **Option 2:** Factor the numerator and denominator:
    - If the factor divides completely from the denominator, then we have a hole at that x-value.
    - If the factor remains in the denominator after dividing common factors, we have a vertical asymptote at that value of x.

1. Evaluate the following limits:

(a) $\lim\limits_{x\to-\infty}(-7x^6 + 2x^3 - 5x + 25)$

leading term

Polynomial ⇒ Look at the leading term

$= \lim\limits_{x\to-\infty} -7x^6 \to \boxed{-\infty}$

multiplying by a neg; large pos #

(b) $\lim\limits_{x\to\infty}(4x^2 - 7x^9 + 5x^3 - 9)$

leading term

$= \lim\limits_{x\to\infty} -7x^9 \to \boxed{-\infty}$

multiplying by a neg; large pos #

2. Determine the end behavior of $f(x) = -3x^2 - Ax^3 - 9$. where A is a real number such that $A < -3$.

leading term

Polynomial ⇒ Look at the leading term

⇒ ① $\lim\limits_{x\to-\infty} f(x)$ & ② $\lim\limits_{x\to\infty} f(x)$

A is negative
−A is positive

① $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} -Ax^3 \to -\infty$ Thus, $\boxed{\lim\limits_{x\to-\infty} f(x) \to -\infty}$

multiplying by a pos.; large neg #

② $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} -Ax^3 \to \infty$

multiplying by a pos.; large pos #

Thus, $\boxed{\lim\limits_{x\to\infty} f(x) \to \infty}$

3. Evaluate the following limits algebraically.

(a) $\lim\limits_{x\to\infty}\dfrac{4x^7 - 3x^2 + 5x}{200 - 9x^2 - 5x^8}$

Rational Function ⇒ Divide every term by the highest power of x in the denominator.

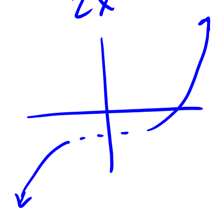
$= \lim\limits_{x\to\infty} \dfrac{\frac{4x^7}{x^8} - \frac{3x^2}{x^8} + \frac{5x}{x^8}}{\frac{200}{x^8} - \frac{9x^2}{x^8} - \frac{5x^8}{x^8}}$

Let's look at the numerator & denominator separately:

$\lim\limits_{x\to\infty}\left(\frac{4}{x} - \frac{3}{x^6} + \frac{5}{x^7}\right) = 0$ $\quad\lim\limits_{x\to\infty}\left(\frac{200}{x^8} - \frac{9}{x^6} - 5\right) = -5$

Thus, $\lim\limits_{x\to\infty}\dfrac{4x^7 - 3x^2 + 5x}{200 - 9x^2 - 5x^8} = \dfrac{0}{-5} = \boxed{0}$

(b) $\lim\limits_{x\to-\infty}\dfrac{2x^6 - 5x}{4 - 5x^3} = \lim\limits_{x\to-\infty}\dfrac{\frac{2x^6}{x^3} - \frac{5x}{x^3}}{\frac{4}{x^3} - \frac{5x^3}{x^3}}$



Let's look at the numerator & denom. separately:

$\lim\limits_{x\to-\infty}\left(2x^3 - \frac{5}{x^2}\right) \to -\infty$ (tends towards $-\infty$; a large neg #)

$\lim\limits_{x\to-\infty}\left(\frac{4}{x^3} - 5\right) = -5$ (dividing by a neg.)

Thus, $\boxed{\lim\limits_{x\to-\infty}\dfrac{2x^6 - 5x}{4 - 5x^3} \to \infty}$

(c) $\lim_{x\to-\infty}\left(\dfrac{3x}{4x^2-7}+\dfrac{5x^3}{Bx^2-15x^3}\right)$ where B is a real number such that $B>0$

$= \lim_{x\to-\infty}\dfrac{3x}{4x^2-7} + \lim_{x\to-\infty}\dfrac{5x^3}{Bx^2-15x^3} = 0 + -\frac{1}{3} = \boxed{-\frac{1}{3}}$

Rational Function ⇒ Divide every term by the highest power in the denom.

① $\lim_{x\to-\infty}\dfrac{3x}{4x^2-7} = \lim_{x\to-\infty}\dfrac{\frac{3x}{x^2}}{\frac{4x^2}{x^2}-\frac{7}{x^2}}$

$\lim_{x\to-\infty}\frac{3}{x} = 0$

$\lim_{x\to-\infty}\left(4-\frac{7}{x^2}\right) = 4$

$\Rightarrow \lim_{x\to-\infty}\dfrac{3x}{4x^2-7} = \frac{0}{4} = 0$

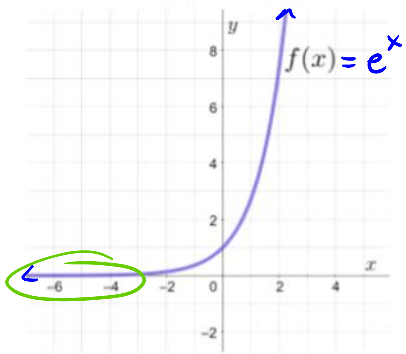
② $\lim_{x\to-\infty}\dfrac{5x^3}{Bx^2-15x^3} = \lim_{x\to-\infty}\dfrac{\frac{5x^3}{x^3}}{\frac{Bx^2}{x^3}-\frac{15x^3}{x^3}}$

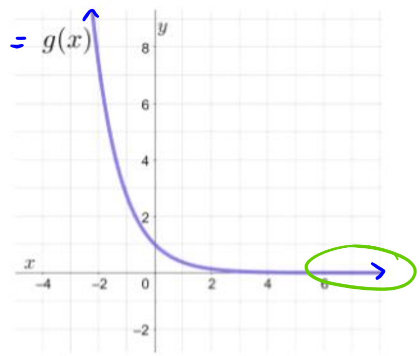
$\lim_{x\to-\infty} 5 = 5$

$\lim_{x\to-\infty}\left(\frac{B}{x}-15\right) = -15$

$\Rightarrow \lim_{x\to-\infty}\dfrac{5x^3}{Bx^2-15x^3} = \frac{5}{-15} = -\frac{1}{3}$

4. Use the graphs of $f(x)$ and $g(x)$ below to determine the following limits.





(a) $\lim_{x\to-\infty} e^x = \boxed{0}$ (b) $\lim_{x\to\infty} e^x \to \boxed{\infty}$ (c) $\lim_{x\to-\infty} e^{-x} \to \boxed{\infty}$ (d) $\lim_{x\to\infty} e^{-x} = \boxed{0}$

5. Evaluate the following limit algebraically.

$$\lim_{x\to-\infty}\frac{4e^{-3x}+3e^x+10}{3e^x+2e^{-4x}}$$

A function involving exponentials ⇒ since $x\to-\infty$, we divide by e^{nx} in the denom where n is most negative

$= \lim_{x\to-\infty}\dfrac{\frac{4e^{-3x}}{e^{-4x}}+\frac{3e^x}{e^{-4x}}+\frac{10e^0}{e^{-4x}}}{\frac{3e^x}{e^{-4x}}+\frac{2e^{-4x}}{e^{-4x}}} = \lim_{x\to-\infty}\dfrac{4e^{-3x-(-4x)}+3e^{x-(-4x)}+10e^{0-(-4x)}}{3e^{x-(-4x)}+2e^{-4x-(-4x)}} = \lim_{x\to-\infty}\dfrac{4e^x+3e^{5x}+10e^{4x}}{3e^{5x}+2}$

$\lim_{x\to-\infty}\left(4e^x+3e^{5x}+10e^{4x}\right) = 0$

$\lim_{x\to-\infty}\left(3e^{5x}+2\right) = 2$

$\Rightarrow \lim_{x\to-\infty}\dfrac{4e^{-3x}+3e^x+10}{3e^x+2e^{-4x}} = \frac{0}{2} = \boxed{0}$

6. Determine all horizontal asymptotes of $f(x) = \dfrac{5e^{-2x} + 4e^{4x}}{-2e^{5x} - 4e^{-2x}}$

① $\lim_{x\to\infty} \dfrac{5e^{-2x}+4e^{4x}}{-2e^{5x}-4e^{-2x}} = \lim_{x\to\infty} \dfrac{\frac{5e^{-2x}}{e^{5x}} + \frac{4e^{4x}}{e^{5x}}}{\frac{-2e^{5x}}{e^{5x}} - \frac{4e^{-2x}}{e^{5x}}} = \lim_{x\to\infty} \dfrac{5e^{-7x}+4e^{-x}}{-2-4e^{-7x}}$

$\lim_{x\to\infty} (5e^{-7x}+4e^{-x}) = 0$

$\lim_{x\to\infty} (-2-4e^{-7x}) = -2$

Thus, $\lim_{x\to\infty} \dfrac{5e^{-2x}+4e^{4x}}{-2e^{5x}-4e^{-2x}} = \dfrac{0}{-2} = 0$

$y=0$ is a horizontal asymptote in the positive infinity direction

② $\lim_{x\to-\infty} \dfrac{5e^{-2x}+4e^{4x}}{-2e^{5x}-4e^{-2x}} = \lim_{x\to-\infty} \dfrac{\frac{5e^{-2x}}{e^{-2x}} + \frac{4e^{4x}}{e^{-2x}}}{\frac{-2e^{5x}}{e^{-2x}} - \frac{4e^{-2x}}{e^{-2x}}} = \lim_{x\to-\infty} \dfrac{5+4e^{6x}}{-2e^{7x}-4}$

$\lim_{x\to-\infty} (5+4e^{6x}) = 5$

$\lim_{x\to-\infty} (-2e^{7x}-4) = -4$

Thus, $\lim_{x\to-\infty} \dfrac{5e^{-2x}+4e^{4x}}{-2e^{5x}-4e^{-2x}} = \dfrac{5}{-4} = -\dfrac{5}{4}$

$y=-\frac{5}{4}$ is a horizontal asymptote in the negative infinity direction

7. Determine all horizontal ② and vertical asymptote(s) ① and hole(s) for $f(x) = \dfrac{(x-3)(x+1)}{x^2(x-4)(x+1)}$. For each vertical asymptote, use limit notation to describe the behavior of the function near the vertical asymptote.

① The only places VA and holes might possibly exist are values of x that are not in the domain of f(x).

$x^2(x-4)(x+1) \neq 0$

$x \neq 0$, $x \neq 4$, $x \neq -1$

Here, the only possibilities are $x=0$, $x=4$ & $x=-1$

$f(x) = \dfrac{(x-3)(x+1)}{x^2(x-4)(x+1)}$ Since $x+1$ cancels completely from the denom:

⇒ There is a hole at $x=-1$.

Since $x^2(x-4)$ remains in the denom ⇒ $x=0$ & $x=4$ are VA

② $f(x) = \dfrac{(x-3)(x+1)}{x^2(x-4)(x+1)} = \dfrac{x^2-2x-3}{x^2(x^2-3x-4)} = \dfrac{x^2-2x-3}{x^4-3x^3-4x^2}$

$\lim_{x\to\pm\infty} \dfrac{x^2-2x-3}{x^4-3x^3-4x^2} = \lim_{x\to\pm\infty} \dfrac{\frac{x^2}{x^4}-\frac{2x}{x^4}-\frac{3}{x^4}}{\frac{x^4}{x^4}-\frac{3x^3}{x^4}-\frac{4x^2}{x^4}} = \lim_{x\to\pm\infty} \dfrac{\frac{1}{x^2}-\frac{2}{x^3}-\frac{3}{x^4}}{1-\frac{3}{x}-\frac{4}{x^2}} = \dfrac{0}{1} = 0$ ⇒ $y=0$ is a HA in both directions

Section 1.4

- A function f is **continuous** at the point $x = c$ if all of the following are true:

  I. $f(c)$ is defined

  II. $\lim_{x\to c} f(x)$ exists

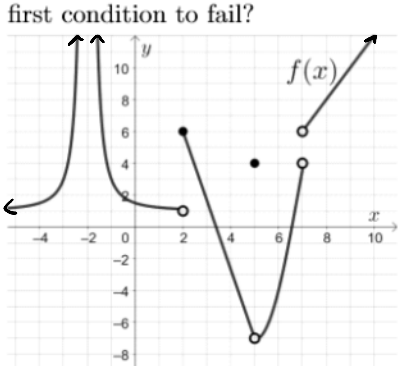
  III. $\lim_{x\to c} f(x) = f(c)$

  If one or more of the conditions are not met, we say f is **discontinuous** at $x = c$.

- Polynomials, exponential functions, logarithmic functions, rational functions, power functions, and combinations of these are continuous on their domain. Thus, to find where these functions are **continuous** we only need to determine the **domain** of the function.

- To determine where a piece-wise defined function is continuous, we must compare the domain restrictions for each rule with the x-values defined for that rule AND check the definition of continuity at each cut-off number.

8. Given the graph of $f(x)$ below, for what value(s) of x is the function discontinuous? What is the first condition to fail?



$x=-2$: I. $f(-2)$ is not defined $\Rightarrow$ I

$x=2$: I. $f(2)$ is defined ✓
II. $\lim_{x\to 2} f(x)$ DNE $\Rightarrow$ II

$x=5$: I. $f(5)$ is defined ✓
II. $\lim_{x\to 5} f(x)$ exist ✓
III. $\lim_{x\to 5} f(x) \neq f(5) \Rightarrow$ III
(-7) (4)

$x=7$: I. $f(7)$ is not defined $\Rightarrow$ I

9. On what interval is each function continuous? Give your answer using interval notation.

   • For (a)-(d), our functions are continuous on their domain. Thus, we just need to find the domain of each function.

   • Recall our domain restrictions:
   ① $\frac{f(x)}{g(x)} \rightarrow g(x) \neq 0$
   ② $\sqrt[n]{f(x)}$ where n is even $\rightarrow f(x) \geq 0$
   ③ $\log_b(f(x))$ for any base $b \rightarrow f(x) > 0$

   (a) $f(x) = 4x^2 - 7x + 2$

   Poly $\Rightarrow$ Domain is $(-\infty, \infty)$

   (b) $g(x) = \dfrac{2x - 7}{x^2 + 4x - 21}$

   $x^2+4x-21 \neq 0$
   $(x+7)(x-3) \neq 0$
   $x+7 \neq 0 \quad x-3 \neq 0$
   $x \neq -7 \quad x \neq 3$

   -7 3

   $(-\infty,-7)\cup(-7,3)\cup(3,\infty)$

(c) $v(t) = \dfrac{\sqrt[7]{t-3}}{e^{\frac{t-2}{t}}}$

odd root ⇒ No worries!

① $e^{\frac{t-2}{t}} \neq 0$ ← this is never zero!

① $t \neq 0$

$(-\infty, 0) \cup (0, \infty)$

(d) $h(x) = \dfrac{\sqrt[4]{x-1}}{\ln(-2x+10)}$

① $\ln(-2x+10) \neq 0$

$e^0 \neq -2x+10$

$1 \neq -2x+10$

$-9 \neq -2x$

$9/2 \neq x$

② $x - 1 \geq 0$

$x \geq 1$

③ $-2x + 10 > 0$

$-2x > -10$

$x < 5$

$[1, 9/2) \cup (9/2, 5)$

Note: A rule is only given for $(-\infty, 0)$

(e) $k(x) = \begin{cases} x^2 + 5 & \text{if } x \leq -5 \\ \dfrac{x-2}{x+4} & \text{if } -5 < x < 0 \end{cases}$

① Check each rule (domain):

ⓐ No domain rest.

ⓑ $x + 4 \neq 0$, $x \neq -4$. Since $x = -4$ is on the int. $-5 < x < 0$ ⇒ $k(x)$ is discont. at $x = -4$

② Check each cutoff (Def. of continuity)

$x = -5$

I. $k(-5) = (-5)^2 + 5 = 30$ ✓

II. $\lim_{x \to -5^-} k(x) = \lim_{x \to -5^-} (x^2+5) = 30$

$\lim_{x \to -5^+} k(x) = \lim_{x \to -5^+} \dfrac{x-2}{x+4} = \dfrac{-5-2}{-5+4} = \dfrac{-7}{-1} = 7$

⇒ $\lim_{x \to -5} k(x)$ DNE

⇒ $k(x)$ is discont. at $x = -5$

$k(x)$ is cont. on $(-\infty, -5) \cup (-5, -4) \cup (-4, 0)$

10. Let $g(x) = \dfrac{x^2 - 3x - 10}{2x^2 - 11x + 5}$

(a) Is $g(x)$ continuous at $x = 3$? Use the definition of continuity to justify your answer.

I. $g(3) = \dfrac{3^2 - 3(3) - 10}{2(3)^2 - 11(3) + 5} = \dfrac{-10}{-10} = 1$ ✓

II. $\lim_{x \to 3} g(x) = \dfrac{3^2 - 3(3) - 10}{2(3)^2 - 11(3) + 5} = \dfrac{-10}{10} = 1$ ✓

III. $\lim_{x \to 3} g(x) = g(3)$ ✓ (1 = 1)

Yes, $g(x)$ is cont. at $x = 3$.

(b) Find all value(s) of x for which $g(x)$ is discontinuous.

Not a piecewise function ⇒ Find domain!

$2x^2 - 11x + 5 \neq 0$

$(2x - 1)(x - 5) \neq 0$

$2x - 1 \neq 0$, $x - 5 \neq 0$

$2x \neq 1$, $x \neq 5$

$x \neq 1/2$

$g(x)$ is discontinuous at $x = 1/2$ & $x = 5$

11. Find the value(s) of k that makes $f(x)$ continuous at $x = -3$.

$$f(x) = \begin{cases} kx^2 - 9 & \text{if } x < -3 \\ x^2 - 15 & \text{if } -3 \le x \le 5 \\ \log(x-4) + 10 & \text{if } x > 5 \end{cases}$$

For $f(x)$ to be continuous at $x = -3$, all 3 conditions must be satisfied:

I. $f(-3) = (-3)^2 - 15 = 9 - 15 = -6$ ✓

II. $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (kx^2 - 9) = k(-3)^2 - 9 = 9k - 9$

$\lim_{x \to -3^+} f(x) = \lim_{x \to -5^+} (x^2 - 15) = (-3)^2 - 15 = -6$

For $\lim_{x \to -3} f(x)$ to exist, $\lim_{x \to -3^-} f(x) = \lim_{x \to -5^+} f(x)$

$9k - 9 = -6$

$9k = 3$

$k = 1/3$

III. when $k = 1/3$, $f(-3) = 9(1/3) - 9 = -6$

Thus, $\lim_{x \to -3} f(x) = f(-3)$ ✓ (−6 = −6)

Thus, $f(x)$ is continuous at $x = -3$ when $k = 1/3$.