Math 308: Week-in-Review 7 (3.7-3.8, 6.1-6.2)

3.7-3.8 Mechanical Vibrations

Review

- Standard equation: m is the mass, c is the damping coefficient and k the spring constant

$$mu'' + cu' + ku = F(t)$$

- Amplitude formula: if $u(t) = A\cos(\omega t) + B\sin(\omega t)$ the amplitude R is given by

$$R = \sqrt{A^2 + B^2}$$

- Critical damping:

$$c = 2\sqrt{mk}$$

- Resonance occurs when forcing frequency matches natural frequency

6.1 Laplace Transform

Review

- Definition:

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)dt$$

- Key transforms:

$$\begin{aligned} e^{at} &\Rightarrow \frac{1}{s-a} \\ t^n &\Rightarrow \frac{n!}{s^{n+1}} \\ e^{at}\cos(bt) &\Rightarrow \frac{s-a}{(s-a)^2+b^2} \end{aligned}$$

- Linearity:

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

1. A 2.5 kg mass stretches a spring by 45 cm. With damping coefficient $c = 1.8$ N·s/m, the mass is displaced by 12 cm upwards and released with an initial velocity of 1.2 m/s downwards. Find the position function, frequency, period, and amplitude. ($g = 9.8$ m/s^2 and assume that downward is positive.)

Solution: Find the spring constant k using Hooke's Law at equilibrium:

$$k = \frac{\text{Force}}{\text{extension}} = \frac{mg}{\text{extension}} = \frac{2.5 \cdot 9.8}{0.45} = 54.4 \text{ N/m}$$

Setting up the differential equation:

$$2.5u'' + 1.8u' + 54.4u = 0$$

Initial conditions (upward displacement is negative):

$$u(0) = -0.12 \text{ m}, \quad u'(0) = 1.2 \text{ m/s}$$

Solving the characteristic equation:

$$2.5\lambda^2 + 1.8\lambda + 54.4 = 0$$

Discriminant:

$$\Delta = 1.8^2 - 4(2.5)(54.4) = -541.2 < 0 \Rightarrow \text{Underdamped}$$

Complex roots:

$$r = \frac{-1.8 \pm i\sqrt{541.2}}{5} \approx -0.36 \pm 4.65i$$

General solution:

$$u(t) = e^{-0.36t}[A\cos(4.65t) + B\sin(4.65t)]$$

Apply initial conditions:

$$\begin{aligned} u(0) &= -0.12 = A \\ u'(0) &= 1.2 = -0.36A + 4.65B \\ 1.2 &= -0.36(-0.12) + 4.65B \\ B &\approx 0.25 \end{aligned}$$

Compute vibration characteristics:

$$\begin{aligned} \text{Amplitude at } t = 0, \ \ R &= \sqrt{(-0.12)^2 + (0.25)^2} \approx 0.28 \text{ m} \\ \text{Angular frequency } \omega &= 4.65 \text{ rad/s} \\ \text{Frequency } f &= \frac{\omega}{2\pi} \approx 0.74 \text{ Hz} \ \ (\text{ rotations per second }) \\ \text{Period } T &= \frac{2\pi}{\omega} = \frac{1}{f} \approx 1.35 \text{ s} \end{aligned}$$

$$\boxed{u(t) = e^{-0.36t}\left[-0.12\cos(4.65t) + 0.25\sin(4.65t)\right], \ f \approx 0.74 \text{ Hz}, \ T \approx 1.35 \text{ s}, \ R \approx 0.28 \text{ m}}$$

2. A 7 N weight stretches a spring by 180 cm. When moving at 2.4 m/s, the damping force is 3.6 N. Starting from 20 cm below the equilibrium position with an upward velocity of 1.8 m/s, find the position function. ($g = 9.8$ m/s^2)

Solution:

Determine system parameters:

$$m = \frac{W}{g} = \frac{7}{9.8} \approx 0.714 \text{ kg}$$
$$k = \frac{7}{1.8} \approx 3.889 \text{ N/m}$$
$$c = \frac{3.6}{2.4} = 1.5 \text{ N·s/m}$$

Set up differential equation:

$$0.714u'' + 1.5u' + 3.889u = 0$$

Initial conditions:

$$u(0) = 0.2 \text{ m}, \quad u'(0) = -1.8 \text{ m/s}$$

Solve characteristic equation:

$$0.714\lambda^2 + 1.5\lambda + 3.889 = 0$$

Discriminant:

$$\Delta = 1.5^2 - 4(0.714)(3.889) \approx -8.86 < 0 \Rightarrow \text{Underdamped}$$

Complex roots:

$$\lambda = \frac{-1.5 \pm i\sqrt{8.86}}{1.428} \approx -1.05 \pm 2.49i$$

General solution:

$$u(t) = e^{-1.05t}[A\cos(2.49t) + B\sin(2.49t)]$$

Apply initial conditions:

$$u(0) = 0.2 = A$$
$$u'(0) = -1.8 = -1.05(0.2) + 2.49B$$
$$B \approx -0.64$$

$$\boxed{u(t) = e^{-1.05t}\left[0.2\cos(2.49t) - 0.64\sin(2.49t)\right]}$$

3. A 1 kg mass-spring system with spring constant $k = 9$ N/m experiences forcing $\sin(3t)$. Starting at rest from equilibrium, find the position function assuming no damping. What happens as $t \to \infty$?

Solution:

Set up equation

$$u'' + 9u = \sin(3t), \quad u(0) = 0,\ u'(0) = 0$$

Homogeneous solution

$$u_c = C_1 \cos(3t) + C_2 \sin(3t)$$

Particular solution (Resonance case): Assume form,

$$u_p(t) = t[A\cos(3t) + B\sin(3t)]$$

Compute derivatives:

$$\begin{aligned} u_p' &= A\cos(3t) + B\sin(3t) + t[-3A\sin(3t) + 3B\cos(3t)] \\ u_p'' &= -6A\sin(3t) + 6B\cos(3t) - 9t[A\cos(3t) + B\sin(3t)] \end{aligned}$$

Substitute into DE:

$$(-6A\sin(3t) + 6B\cos(3t)) = \sin(3t)$$

Equate coefficients:

$$\begin{aligned} -6A = 1 &\Rightarrow A = -\frac{1}{6} \\ 6B = 0 &\Rightarrow B = 0 \end{aligned}$$

General solution

$$u(t) = C_1\cos(3t) + C_2\sin(3t) - \frac{t}{6}\cos(3t)$$

Apply initial conditions

$$\begin{aligned} u(0) &= C_1 = 0 \\ u'(0) &= 3C_2 - \frac{1}{6} = 0 \Rightarrow C_2 = \frac{1}{18} \end{aligned}$$

Solution of problem:

$$\boxed{u(t) = \frac{1}{18}\sin(3t) - \frac{t}{6}\cos(3t)}$$

As $t \to \infty$, the $-\dfrac{t}{6}\cos(3t)$ term dominates, causing unbounded growth.

4. For $3u'' + cu' + 15u = 0$, determine the value of c for critical damping.

   Characteristic Equation: For $3u'' + cu' + 15u = 0$, the characteristic equation is:

$$3\lambda^2 + c\lambda + 15 = 0$$

   Discriminant: For critical damping, the discriminant must equal zero:

$$\Delta = c^2 - 4(3)(15) = c^2 - 180 = 0$$

   Solve for c:

$$c^2 = 180 \quad \Rightarrow \quad c = \sqrt{180} = 6\sqrt{5}$$

$$\boxed{c = 6\sqrt{5}}$$

6.1: DEFINITION OF LAPLACE TRANSFORM

Review

- The Laplace transform is defined by

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

- For many functions, you can just look up the Laplace transform in the table.

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at} t^n (n = 1, 2, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

- The Laplace transform is also linear:

$$\mathcal{L}\{c_1 f + c_2 g\} = c_1 \mathcal{L}\{f\} + c_2 \mathcal{L}\{g\}$$

- To take the inverse Laplace transform, you can also use the table. However, if your function does not match the things in the table, then you need to first do partial fractions.
- Partial fractions review
  - Simple roots
  - Irreducible quadratics
  - Repeated roots

5. Compute

$$\mathcal{L}\{5t^2\}$$

using integration by parts.

Solution: Apply the definition of Laplace Transform:

$$\mathcal{L}\{5t^2\} = 5\int_0^\infty t^2 e^{-st} dt$$

Integration by parts: Let $u = t^2$ and $dv = e^{-st}dt$:

$$\begin{aligned}\int t^2 e^{-st} dt &= -\frac{t^2}{s}e^{-st}\Big|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}dt \\ &= 0 + \frac{2}{s}\mathcal{L}\{t\}\end{aligned}$$

Integrating by parts again: Let $u = t$ and $dv = e^{-st}dt$:

$$\begin{aligned}\int te^{-st} dt &= -\frac{t}{s}e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}dt \\ &= 0 - \frac{1}{s^2}e^{-st}\Big|_0^\infty\end{aligned}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \implies \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}$$

Combining results:

$$\mathcal{L}\{5t^2\} = 5\cdot\frac{2}{s^3} = \frac{10}{s^3}$$

$$\boxed{\mathcal{L}\{5t^2\} = \frac{10}{s^3}}$$

6. Find $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ e^{-t} & t \geq 4 \end{cases}$.

Solution: Split integral

$$\mathcal{L}\{f(t)\} = \int_0^4 2e^{-st}dt + \int_4^\infty e^{-t}e^{-st}dt$$

Evaluate first integral:

$$2\int_0^4 e^{-st}dt = -\frac{2}{s}e^{-st}\bigg|_0^4 = \frac{2}{s}(1 - e^{-4s})$$

Evaluate second integral:

$$\int_4^\infty e^{-(s+1)t}dt = -\frac{1}{(s+1)}e^{-(s+1)t}\bigg|_4^\infty = \frac{e^{-4(s+1)}}{s+1}$$

Combining results:

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s}(1 - e^{-4s}) + \frac{e^{-4(s+1)}}{s+1}}$$

7. Compute $\mathcal{L}\{3t^5 - 4\cos(\pi t) + e^{-2t}\sin(3t)\}$.

Solution: Apply linearity:

$$\mathcal{L}\{3t^5 - 4\cos(\pi t) + e^{-2t}\sin(3t)\} = 3\mathcal{L}\{t^5\} - 4\mathcal{L}\{\cos(\pi t)\} + \mathcal{L}\{e^{-2t}\sin(3t)\}$$

Use transform table:

$$\begin{aligned} \mathcal{L}\{t^5\} &= \frac{5!}{s^6} = \frac{120}{s^6} \\ \mathcal{L}\{\cos(\pi t)\} &= \frac{s}{s^2+\pi^2} \\ \mathcal{L}\{e^{-2t}\sin(3t)\} &= \frac{3}{(s+2)^2+9} \end{aligned}$$

Combining results

$$\boxed{\mathcal{L}\{f(t)\} = \frac{360}{s^6} - \frac{4s}{s^2+\pi^2} + \frac{3}{(s+2)^2+9}}$$

8. Find $\mathcal{L}^{-1}\left\{\frac{4}{s^6}\right\}$.

Solution:

Recognize standard form:

$$\frac{4}{s^6} = 4 \cdot \frac{1}{s^{5+1}} = 4 \cdot \frac{5!}{5!s^6} = \frac{4}{5!} \cdot \frac{5!}{s^6}$$

Apply inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^6}\right\} = \frac{4}{5!} \cdot \mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} = \frac{4}{120} \cdot t^5 = \frac{t^5}{30}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{4}{s^6}\right\} = \frac{t^5}{30}}$$

9. Determine $\mathcal{L}^{-1}\left\{\dfrac{5s}{s^2+5}\right\}$.

Solution: Adjust to standard form

$$\frac{5s}{s^2+5} = 5\cdot\frac{s}{s^2+(\sqrt{5})^2}$$

Apply inverse cosine transform:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos(bt)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{5s}{s^2+5}\right\} = 5\cos(\sqrt{5}t)}$$

10. Compute $\mathcal{L}^{-1}\left\{\dfrac{3}{(s+1)^2+9}\right\}$.

Solution:

Recognize exponential shift

$$\frac{3}{(s+1)^2+9} = \frac{3}{(s-(-1))^2+3^2}$$

Apply inverse sine transform

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin(bt)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\} = e^{-t}\sin(3t)}$$

11. Solve $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{(s+2)(s^2+4s+8)}\right\}$

Solution: Factor denominator:

$$s^2+4s+8=(s+2)^2+4$$

Partial fraction decomposition:

$$\frac{2s-1}{(s+2)[(s+2)^2+4]}=\frac{A}{s+2}+\frac{B(s+2)+C}{(s+2)^2+4}$$

Solve for coefficients Multiply through and equate:

$$2s-1=A[(s+2)^2+4]+[B(s+2)+C](s+2)$$

Let $s=-2$:

$$-5=4A\Rightarrow A=-\frac{5}{4}$$

Expanding and solving system (choose $s=0, s=-1$ for example):

$$B=\frac{5}{4},\ C=2$$

Apply inverse transforms:

$$\mathcal{L}^{-1}\left\{\frac{-5/4}{s+2}\right\}=-\frac{5}{4}e^{-2t}$$
$$\mathcal{L}^{-1}\left\{\frac{5/4(s+2)}{(s+2)^2+4}\right\}=\frac{5}{4}e^{-2t}\cos(2t)$$
$$\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\}=e^{-2t}\sin(2t)$$

$$\boxed{\mathcal{L}^{-1}=-\frac{5}{4}e^{-2t}+\frac{5}{4}e^{-2t}\cos(2t)+e^{-2t}\sin(2t)}$$

12. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s-3)^3(s+1)}\right\}$.

Solution: Partial fraction decomposition

$$\frac{s^2}{(s-3)^3(s+1)} = \frac{A}{s+1} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3}$$

Solve for coefficients: Multiply through and solve:

$$s^2 = A(s-3)^3 + B(s+1)(s-3)^2 + C(s+1)(s-3) + D(s+1)$$

Let $s=-1$:

$$1 = -64A \Rightarrow A = -\frac{1}{64}$$

Let $s=3$:

$$9 = 4D \Rightarrow D = \frac{9}{4}$$

Solving remaining system gives:

$$B = \frac{1}{64},\ C = -\frac{15}{16}$$

Apply inverse transforms:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{-1/64}{s+1}\right\} &= -\frac{1}{64}e^{-t} \\
\mathcal{L}^{-1}\left\{\frac{1/64}{s-3}\right\} &= \frac{1}{64}e^{3t} \\
\mathcal{L}^{-1}\left\{\frac{-15/16}{(s-3)^2}\right\} &= -\frac{15}{16}te^{3t} \\
\mathcal{L}^{-1}\left\{\frac{9/4}{(s-3)^3}\right\} &= \frac{9}{8}\frac{t^2}{2}e^{3t}
\end{aligned}$$

$$\boxed{\mathcal{L}^{-1} = -\frac{1}{64}e^{-t} + \frac{1}{64}e^{3t} - \frac{15}{16}te^{3t} + \frac{9}{8}t^2e^{3t}}$$