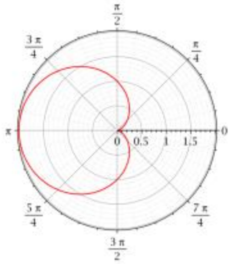
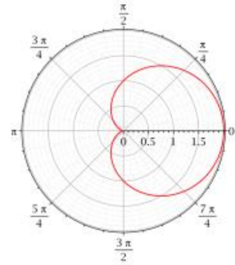
Curves in Polar Coordinates

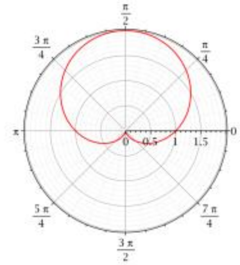
$r = 1 - \cos(\theta)$



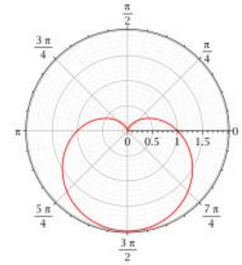
$r = 1 + \cos(\theta)$



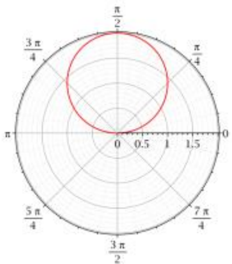
$r = 1 + \sin(\theta)$



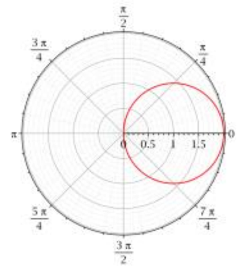
$r = 1 - \sin(\theta)$



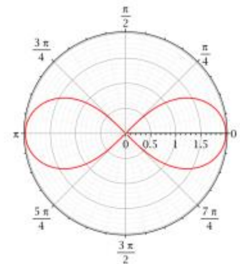
$r = 2\sin(\theta)$



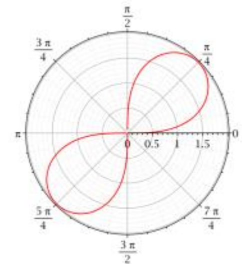
$r = 2\cos(\theta)$



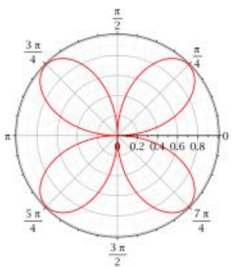
$r^2 = \cos(2\theta)$



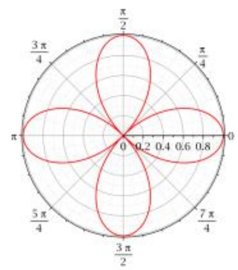
$r^2 = \sin(2\theta)$



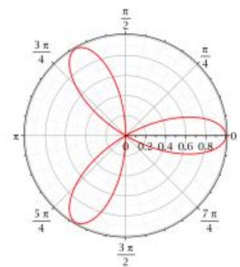
$r = \sin(2\theta)$



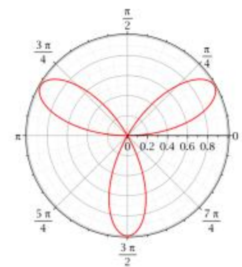
$r = \cos(2\theta)$



$r = \cos(3\theta)$



$r = \sin(3\theta)$



$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$ $\qquad$ $\iint_R f(x)\cdot g(y)\,dA = \int_a^b f(x)\,dx \int_c^d g(y)\,dy$

$R = \underbrace{[a,b]}_{x} \times \underbrace{[c,d]}_{y}$ $\qquad$ $R=[a,b]\times[c,d]$ works only if R is a rectangle

1. Find the integral $\iint_R \frac{y\cos y}{x}\,dA$, where $R = \{(x,y)|\,1 \le x \le e^4, 0 \le y \le \pi/2\}$.

$$= \int_0^{\pi/2} \left[\int_1^{e^4} \frac{y\cos y}{x}\,dx \right] dy = \underbrace{\int_0^{\pi/2} y\cos y\,dy}_{\text{by parts}} \int_1^{e^4} \frac{1}{x}\,dx$$

	D	I
+	y	cos y
−	1	sin y
	0	− cos y

$$= \left(y\sin y + \cos y\right)\Big|_0^{\pi/4} \cdot \Big[\ln|x|\Big]_1^{e^4}$$

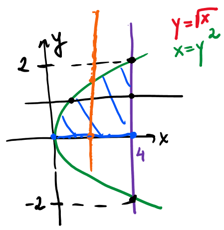
$$= \left(\frac{\pi}{4}\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \cos 0\right)\left(\ln e^4 - \ln 1\right)$$

$\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$, 1, 4, 0

$$= \boxed{4\left(\frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2} - 1\right)}$$

2. Evaluate $\iint_D \frac{y}{\sqrt{1+x^2}}\, dA$ where D is the region in the first quadrant bounded by $x = y^2$, $x = 4$, $y = 0$.

x-axis.



Point of intersection

$x = y^2$ and $x = 4$

$y^2 = 4$, $y = \pm 2$

$dx\, dy$

$y^2 \le x \le 4$

[parabola] $x = y^2$ [the line $x = 4$]

$0 \le y \le 2$

[min value] [max value]

$$\int_0^2 \int_{y^2}^4 \frac{y}{\sqrt{1+x^2}}\, dx\, dy$$

$$= \int_0^2 y \arcsin(x) \Big|_{x=y^2}^{x=4} dy$$

$$= \int_0^2 y \left[\arcsin 4 - \arcsin(y^2)\right] dy$$

$\boxed{\text{way too complicated}}$

$dy\, dx$

[x-axis] $\le y \le$ [parabola]

[curve on the bottom] [curve on the top]

$0 \le y \le \sqrt{x}$

$0 \le x \le 4$

[min value] [max value]

$$\int_0^4 \int_0^{\sqrt{x}} \frac{y}{\sqrt{1+x^2}}\, dy\, dx$$

$$= \int_0^4 \frac{y^2}{2} \Big]_0^{\sqrt{x}} \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \int_0^4 \frac{x}{2} \cdot \frac{1}{\sqrt{1+x^2}}\, dx$$

$u = 1 + x^2$

$du = 2x\, dx \Rightarrow x\, dx = \frac{du}{2}$

$x = 0: \; u(0) = 1 + 0^2 = 1$

$x = 4: \; u(4) = 1 + 4^2 = 17$

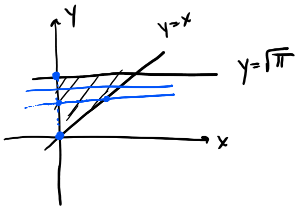
$$= \frac{1}{2}\int_1^{17} \frac{\frac{du}{2}}{\sqrt{u}}$$

$$= \frac{1}{4} \frac{u^{1/2}}{1/2} \Big|_1^{17} = \frac{2}{4}\left(\sqrt{17} - \sqrt{1}\right)$$

$$= \boxed{\frac{1}{2}\left(\sqrt{17} - 1\right)}$$

$$\boxed{\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C}$$

3. Evaluate $\displaystyle\int\int_R y^2 \sin\frac{xy}{2}\,dA$ where R is the region bounded by $\underbrace{x = 0}_{y\text{-axis}}$, $y = \sqrt{\pi}$, $y = x$.



$dx\,dy$

$[y\text{-axis}] \le x \le [y=x]$

$0 \le x \le y$

$0 \le y \le \sqrt{\pi}$

$$= \int_0^{\sqrt{\pi}} \int_0^{y} y^2 \sin\frac{xy}{2}\,dx\,dy = \int_0^{\sqrt{\pi}} y^2 \left[\left(-\cos\frac{xy}{2}\right)\cdot\frac{2}{y}\right]_0^{y} dy$$

$$= -\int_0^{\sqrt{\pi}} 2y \left[-\cos\left(\frac{y^2}{2}\right) + \overset{1}{\cancel{\cos 0}}\right] dy$$

$$= -\int_0^{\sqrt{\pi}} \left[2y - 2y\cos\left(\frac{y^2}{2}\right)\right] dy = -\int_0^{\sqrt{\pi}} 2y\,dy + \int_0^{\sqrt{\pi}} 2y\cos\left(\frac{y^2}{2}\right) dy$$

$u = \frac{y^2}{2}$, $du = y\,dy$

$y = 0 \longrightarrow u = \frac{0}{2} = 0$

$y = \sqrt{\pi} \longrightarrow u = \frac{\pi}{2}$

$$= -y^2\Big|_0^{\sqrt{\pi}} + 2\int_0^{\pi/2} \cos u\,du = -\pi + 2\int_0^{\pi/2} \cos u\,du$$

$$= -\pi + 2\sin u\Big|_0^{\pi/2} = -\pi + \overset{1}{\cancel{2\sin\frac{\pi}{2}}} - \overset{0}{\cancel{2\sin 0}}$$

$$= -\pi + 2$$

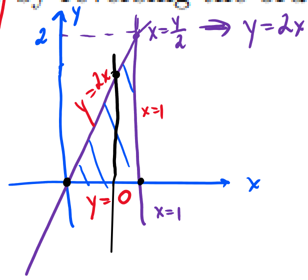
4. Evaluate $\boxed{\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy}$ by reversing the order of integration.

$\frac{y}{2} \le x \le 1$

$0 \le y \le 2$

$x = y/2, \quad x = 1$

$0 \le y \le 2$



$dy\,dx$

$0 \le y \le 2x$

$0 \le x \le 1$

$= \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1 e^{x^2} y \Big|_0^{2x} dx = \int_0^1 2x e^{x^2}\,dx$

$u = x^2$

$du = 2x\,dx$

$x = 0 \to u = 0^2 = 0$

$x = 1 \to u = 1^2 = 1$

$= \int_0^1 e^u\,du = e^u \Big|_0^1 = e^1 - e^0 = \boxed{e-1}$

5. Evaluate $\int_0^1 \int_{2x^2}^2 x^3 \sin(y^3)\,dy\,dx$.

Reverse the order of integration.

$2x^2 \le y \le 2$

$0 \le x \le 1$

$y = 2x^2$

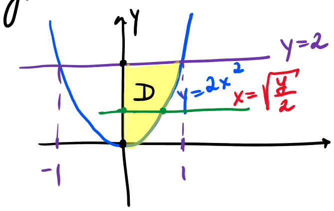
$y = 2$

intersection

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$



dx dy

[y-axis] $\le x \le$ [parabola]

$0 \le x \le \sqrt{\frac{y}{2}}$

$0 \le y \le 2$

$$= \int_0^2 \int_0^{\sqrt{\frac{y}{2}}} x^3 \sin(y^3)\,dx\,dy$$

$$= \int_0^2 \sin(y^3)\, \frac{x^4}{4}\Big|_0^{\sqrt{\frac{y}{2}}} dy = \frac{1}{4}\int_0^2 \sin(y^3)\left(\frac{y}{2}\right)^2 dy = \frac{1}{16}\int_0^2 y^2 \sin(y^3)\,dy$$

$u = y^3$

$du = 3y^2 dy \rightarrow y^2 dy = \frac{du}{3}$

$y = 0 \rightarrow u = 0^3 = 0$

$y = 2 \rightarrow u = 2^3 = 8$

$$= \frac{1}{16}\int_0^8 \sin u \,\frac{du}{3} = \frac{1}{48}\left(-\cos u\right)_0^8 = \frac{1}{48}\left(-\cos 8 + \cancel{\cos 0}^{\,1}\right)$$

$$= \boxed{\frac{1}{48}\left(1 - \cos 8\right)}$$

6. Graph the region and change the order of integration.

a) $\int_0^1 \int_0^{x^3} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2-x} f(x,y)\,dydx = \int_0^1 \int_{\sqrt[3]{y}}^{2-y} f(x,y)\,dx\,dy$

b) $\int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dxdy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} f(x,y)\,dxdy = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy\,dx$

(a) $dy\,dx$

$0 \le x \le 1$

$0 \le y \le x^3$

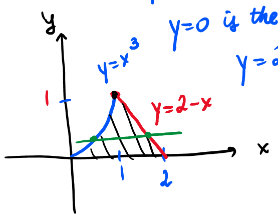
$y = x^3$

$1 \le x \le 2$

$0 \le y \le 2-x$

$y=0$ is the x-axis.

$y = 2-x$



$dx\,dy$

[parabola] $\le x \le$ [line]

$y = x^3$ $\quad$ $y = 2-x$

$x = \sqrt[3]{y}$ $\quad$ $x = 2-y$

$\sqrt[3]{y} \le x \le 2-y$

$0 \le y \le 1$

$dx\,dy$

$0 \le y \le 1$

$0 \le x \le \sqrt{y}$

$x = 0$ $\quad$ y axis

$x = \sqrt{y}$ or $y = x^2$

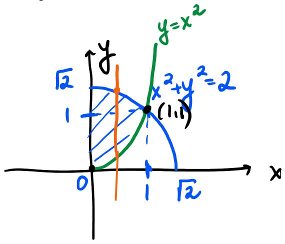
$1 \le y \le \sqrt{2}$

$0 \le x \le \sqrt{2-y^2}$

$(x)^2 = \left(\sqrt{2-y^2}\right)^2$

$x^2 = 2-y^2$

$x^2+y^2 = 2 \leftarrow$ circle centered @ $(0,0)$ of radius $\sqrt{2}$



point of intersection.

$y = x^2$

$x^2+y^2 = 2$

$y + y^2 = 2$

$y^2+y-2=0$

$(y+2)(y-1)=0$

$y_1 = -2$, $y_2 = 1$ $\Rightarrow y = x^2 \Rightarrow x = 1$

ignore

$dy\,dx$

[parabola] $\le y \le$ [circle]

$y = x^2$ $\quad$ $x^2+y^2=2$

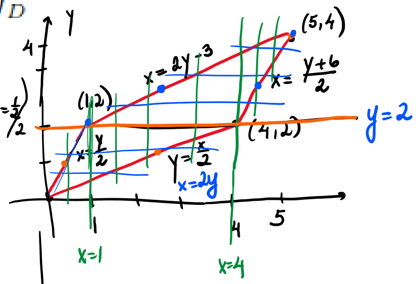
$y^2 = 2-x^2$

$y = \sqrt{2-x^2}$ [top half circle]

$\boxed{x^2 \le y \le \sqrt{2-x^2}}$

$\boxed{0 \le x \le 1}$

7. Let the region D be the parallelogram with the vertices $(0,0)$, $(1,2)$, $(5,4)$, and $(4,2)$. Write the double integral $\iint_D f(x,y)\,dA$ as a sum of iterated integrals (with the least number of terms).



$(1,2) \to (5,4)$

slope $= \frac{4-2}{5-1} = \frac{2}{4} = \frac{1}{2}$

$y-2 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}x - \frac{1}{2} + 2$

$\boxed{y = \frac{1}{2}x + \frac{3}{2}}$

$(4,2) \to (5,4)$

slope $= \frac{4-2}{5-4} = 2$

$y-2 = 2(x-4)$

$y-2 = 2x-8$

$y = 2x-6$

$x = \frac{y+6}{2}$

$dx\,dy \to$ two integrals. ✓

$dy\,dx \to$ three integrals.

$dx\,dy$

$0 \le y \le 2$

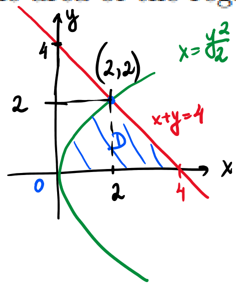
$\frac{y}{2} \le x \le 2y$

$2 \le y \le 4$

$2y-3 \le x \le \frac{y+6}{2}$

$$\iint_D f(x,y)\,dA = \int_0^2 \int_{y/2}^{2y} f(x,y)\,dx\,dy + \int_2^4 \int_{2y-3}^{\frac{y+6}{2}} f(x,y)\,dx\,dy$$

8. Sketch the region bounded by $y^2 = 2x$ (or $x = \frac{y^2}{2}$), the line $x + y = 4$ and the x-axis, in the first quadrant. Find the area of the region using a double integral.



point of intersection

$$\left.\begin{array}{l} x = \frac{y^2}{2} \\ x+y=4 \end{array}\right\} \Rightarrow \frac{y^2}{2}+y=4$$

$$y^2+2y=8$$

$$y^2+2y-8=0$$

$$(y+4)(y-2)=0.$$

$\cancel{y_1=-4}$, $y=2$.

$x = \frac{y^2}{2} = \frac{2^2}{2} = 2.$

$dx\,dy$

$\frac{y^2}{2} \le x \le 4-y$

$0 \le y \le 2$

$dy\,dx$

$0 \le x \le 2$ $2 \le x \le 4$

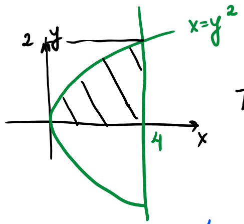
$0 \le y \le \sqrt{2x}$ $0 \le y \le 4-x$

$$A = \iint_D 1\,dA = \int_0^2\int_{\frac{y^2}{2}}^{4-y} dx\,dy = \int_0^2 x\Big|_{\frac{y^2}{2}}^{4-y} dy = \int_0^2\left(4-y-\frac{y^2}{2}\right)dy$$

$$= \left(4y-\frac{y^2}{2}-\frac{y^3}{6}\right)_0^2 = 8-2-\frac{8}{6} = \ldots$$

$$\text{surface } z=x^2+y^2$$

9. Describe the solid which volume is given by the integral $\boxed{V=\int_0^2\int_{y^2}^4 (x^2+y^2)\,dx\,dy}$ and find the volume.



The solid under the paraboloid $z=x^2+y^2$ above the region D: $\boxed{\begin{array}{l} y^2\le x\le 4 \\ 0\le y\le 2\end{array}}$

$$V=\int_0^2\int_{y^2}^4 (x^2+y^2)\,dx\,dy=\int_0^2\left(\frac{x^3}{3}+y^2x\right)\Big|_{y^2}^4 dy$$

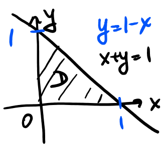
$$=\int_0^2\left[\frac{64}{3}-\frac{(y^2)^3}{3}+4y^2-y^2(y^2)\right]dy$$

$$=\int_0^2\left(\frac{64}{3}-\frac{y^6}{3}+4y^2-y^4\right)dy$$

$$=\left(\frac{64}{3}y-\frac{y^7}{21}+\frac{4y^3}{3}-\frac{y^5}{5}\right)\Big|_0^2=\frac{64\cdot 2}{3}-\frac{128}{21}+\frac{4\cdot 8}{3}-\frac{32}{5}$$

$$=\frac{128}{3}-\frac{128}{21}+\frac{32}{3}-\frac{32}{5}=\ldots$$

10. Find the volume of the solid bounded by
$z = 1 + x + y,\ z = 0,\ \underbrace{x + y = 1, \overbrace{x = 0}^{y\text{-axis}}, \overbrace{y = 0}^{x\text{-axis}}}_{D}.$



dy dx

$$\begin{bmatrix} 0 \le y \le 1-x \\ 0 \le x \le 1 \end{bmatrix}$$

$$V = \iint_D (1+x+y)\, dA$$

$$= \int_0^1 \int_0^{1-x} (1+x+y)\, dy\, dx$$

$$= \int_0^1 \left(y + xy + \frac{y^2}{2} \right)\Big|_0^{1-x} dx$$

$$= \int_0^1 \left(1-x + x(1-x) + \frac{(1-x)^2}{2} \right) dx$$

$$= \int_0^1 \left(1 - \cancel{x} + \cancel{x} - x^2 + \frac{1-2x+x^2}{2} \right) dx$$

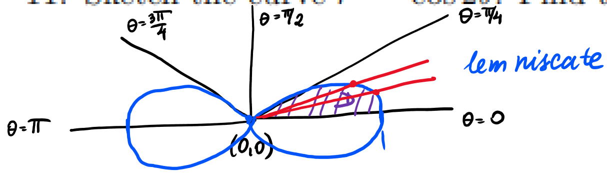
$$= \int_0^1 \left(1 - x^2 + \frac{1}{2} - x + \frac{x^2}{2} \right) dx$$

$$= \int_0^1 \left(\frac{3}{2} - x - \frac{x^2}{2} \right) dx = \left(\frac{3}{2}x - \frac{x^2}{2} - \frac{x^3}{6} \right)\Big|_0^1$$

$$= \frac{3}{2} - \frac{1}{2} - \frac{1}{6} = 1 - \frac{1}{6} = \boxed{\frac{5}{6}}$$

$$\iint_D f(x,y)\,dA \quad \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \end{array} \right|$$

11. Sketch the curve $r^2 = \cos 2\theta$. Find the area inside the curve.



$r^2 = \cos 2\theta$

$\theta = 0 \Rightarrow r^2 = \cos 0 = 1$

$\theta = \frac{\pi}{4} \Rightarrow r^2 = \cos\frac{2\pi}{4} = \cos\frac{\pi}{2} = 0$

$\theta = \frac{\pi}{2} \Rightarrow r^2 = \cos\frac{2\pi}{2} = \cos\pi = -1 \leftarrow$ makes no sence.

$\theta = \frac{3\pi}{4} \Rightarrow r^2 = \cos\frac{6\pi}{4} = \cos\frac{3\pi}{2} = 0$

$\theta = \pi \Rightarrow r^2 = \cos 2\pi = 1$

$A = 4\iint_D r\,dr\,d\theta$

$0 \le r \le \sqrt{\cos 2\theta}$

$0 \le \theta \le \frac{\pi}{4}$

$= 4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta$

$= 4\int_0^{\pi/4} \frac{r^2}{2}\Big|_0^{\sqrt{\cos 2\theta}}\,d\theta$

$= 2\int_0^{\pi/4} \cos 2\theta\,d\theta$

$= 2\,\frac{1}{2}\sin 2\theta\Big|_0^{\pi/4}$

$= \sin 2\theta\Big|_0^{\pi/4}$

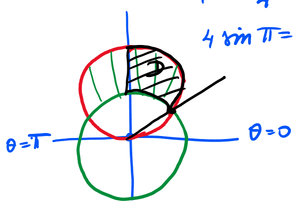
$= \sin\frac{\pi}{2} - \sin 0 = 1$

12. Use a double integral in polar coordinates to evaluate the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.

$r = 4\sin\theta$

$\theta = \frac{\pi}{2}$ $\quad 4\sin\frac{\pi}{4} = 4$

$4\sin\pi = 0.$



intersection

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$A = 2A(D) = 2\iint_D r\,dr\,d\theta$$

$$D: \quad 2 \le r \le 4\sin\theta$$
$$\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$$

$$= 2\int_{\pi/6}^{\pi/2}\int_{2}^{4\sin\theta} r\,dr\,d\theta$$

$$= \cancel{2}\int_{\pi/6}^{\pi/2} \frac{r^2}{\cancel{2}}\Big|_2^{4\sin\theta}\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left(16\sin^2\theta - 4\right)d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \int_{\pi/6}^{\pi/2}\left(8 - 8\cos 2\theta - 4\right)d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left(4 - 8\cos 2\theta\right)d\theta$$

$$= \left(4\theta - \frac{8}{2}\sin 2\theta\right)\Big|_{\pi/6}^{\pi/2}$$

$$= 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right)^{\pi/3} - 4\,\cancel{\sin\pi}^{\,0} + 4\,\cancel{\sin\frac{\pi}{3}}^{\,\frac{\sqrt{3}}{2}}$$

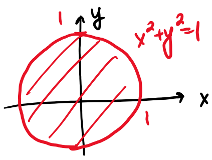
$$= \frac{4\pi}{3} + 4\cdot\frac{\sqrt{3}}{2} = \frac{4\pi}{3} + 2\sqrt{3}$$

13. Use polar coordinates to evaluate

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \longrightarrow x^2 = 1-y^2$ or $x^2+y^2=1$

$-1 \le y \le 1$



polar coordinates

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right]$ $x^2+y^2=r^2$

$dA = r\,dr\,d\theta$

$x^2+y^2=1 \longrightarrow r=1$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi}\int_0^1 \ln(r^2+1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\ln(r^2+1)\,dr$$

$t = r^2+1$

$dt = 2r\,dr \Rightarrow r\,dr = \frac{dt}{2}$

$r=0 \rightarrow t = 0^2+1 = 1$

$r=1 \rightarrow t = 1^2+1 = 2$

$$= 2\pi\int_1^2 \frac{1}{2}\ln t\,dt = \pi\int_1^2 \ln t\,dt$$

by parts:

$u = \ln t \qquad v' = 1$

$u' = \frac{1}{t} \qquad v = t$

$\int uv'\,dt = uv - \int vu'\,dt$

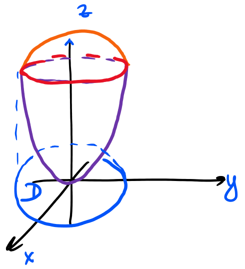
$$= \pi\left[t\ln t\Big|_1^2 - \int_1^2 t\cdot\frac{1}{t}\,dt \right]$$

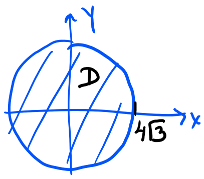
$$= \pi\left[2\ln 2 - 1\,\cancel{\ln 1}^{\,0} - \int_1^2 dt \right]$$

$$= \pi\left(2\ln 2 - t\Big|_1^2\right) = \pi(2\ln 2 - 1)$$

14. Find the volume of the solid bounded by the surfaces

$$\overset{\text{sphere}}{z = \sqrt{64 - x^2 - y^2}} \text{ and } \overset{\text{paraboloid}}{z = \frac{1}{12}(x^2 + y^2)}$$





$0 \le \theta \le 2\pi$

$0 \le r \le 4\sqrt{3}$

curve of intersection:

$$\sqrt{64-x^2-y^2} = \frac{1}{12}\underbrace{(x^2+y^2)}_{r^2}$$

$$\left(\sqrt{64-r^2}\right)^2 = \left(\frac{r^2}{12}\right)^2$$

$$64 - r^2 = \frac{r^4}{144}$$

$$r^4 = 144(64 - r^2)$$

$$r^4 + 144r^2 - 9216 = 0$$

$t \ge 0$ $\quad r^2 = t \Rightarrow r^4 = t^2$

$$t^2 + 144t - 9216 = 0$$

$$t_1 = \frac{-144 + \sqrt{144^2 + 4(9216)}}{2} = \frac{-144 + 240}{2} = 48$$

$$t_2 = \frac{-144 - \sqrt{144^2 + 4(9216)}}{2} = -\frac{144 + 240}{2} < 0 \quad \text{not valid.}$$

$$r^2 = 48 \Rightarrow r = \sqrt{48} = 4\sqrt{3}$$

$$V = \iint_D \left[\sqrt{64 - x^2 - y^2} - \frac{1}{12}(x^2 + y^2)\right] dA$$

$$\left[\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \\ dA = r\,dr\,d\theta \end{array}\right]$$

$$= \iint_D \left(\sqrt{64 - r^2} - \frac{1}{12} r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{4\sqrt{3}} \left[r\sqrt{64 - r^2} - \frac{r^3}{12}\right] dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \left[\int_0^{4\sqrt{3}} r\sqrt{64 - r^2}\,dr - \int_0^{4\sqrt{3}} \frac{r^3}{3}\,dr\right]$$

$u = 64 - r^2$

$du = -2r\,dr \Rightarrow r\,dr = -\frac{du}{2}$

$r = 0 \rightarrow u = 64$

$r = 4\sqrt{3} \rightarrow u = 64 - 48 = 16$

$$= 2\pi \left[-\frac{1}{2}\int_{64}^{16} \sqrt{u}\,du - \frac{r^4}{12}\Big|_0^{4\sqrt{3}}\right]$$

$$= 2\pi \left(-\frac{1}{2} \frac{u^{3/2}}{3/2}\Big|_{64}^{16} - \frac{48^2}{12}\right)$$

$$= 2\pi \left(-\frac{1}{3}\left(-64^{3/2} + 16^{3/2}\right) - \frac{48^2}{12}\right)$$

$$= 2\pi \left(-\frac{1}{3}(-512 + 64) - 192\right)$$