Math 308: Week-in-Review 14 (FINAL EXAM REVIEW)

1. Find the solution. Where is the solution defined?

linear, separable, 1^{st} order

$$y' - x^2 y = x^2, \quad y(0) = 3.$$

Method #1

Linear, 1^{st} order $\Rightarrow$ method of integrating factors

Find $\mu(x)$:

$\mu(x) = e^{\int p(x)dx}$ where $y' + p(x)y = g(t)$, $p(x) = -x^2$

$= e^{\int -x^2 dx} = e^{-x^3/3}$

Multiply by $\mu(x)$

$\underbrace{e^{-x^3/3} y' - e^{-x^3/3} x^2 y}_{\text{reverse product rule follows}} = e^{-x^3/3} x^2$

$\left(e^{-x^3/3} y\right)' = e^{-x^3/3} \cdot x^2$

Integrate

$e^{-x^3/3} y = \int e^{-x^3/3} \cdot x^2 \, dx$

substitution
$u = x^3/3$
$\Rightarrow du = x^2 dx$

$= \int e^{-u} du$

$= -e^{-u} + C = -e^{-x^3/3} + C \Rightarrow y(x) = -1 + Ce^{x^3/3}$

Find C

$y(0) = -1 + C = 3 \Rightarrow C = 4$

$y(x) = -1 + 4e^{x^3/3}$

Method #2

linear, separable

$$y' - x^2 y = x^2 \Rightarrow y' = x^2(1+y)$$

$$\frac{dy}{dx} = x^2(1+y) \Rightarrow \frac{1}{1+y}\,dy = x^2\,dx \Rightarrow \int \frac{1}{1+y}\,dy = \int x^2\,dx$$

$$\ln|1+y| = \frac{x^3}{3} + c_1 \Rightarrow e^{\ln|1+y|} = e^{x^3/3 + c_1} = e^{c_1} \cdot e^{x^3/3}$$

$$|1+y| = C_2 e^{x^3/3}, \quad \text{where } C_2 = e^{c_1}$$

$$1+y = \pm C_2 e^{x^3/3} = C_3 e^{x^3/3}, \quad \text{where } C_3 = \pm C_2$$

$$y = -1 + C_3 e^{x^3/3}$$

Find C_3

$$y(0) = -1 + C_3 e^{0^3/3} = -1 + C_3 = 3 \Rightarrow C_3 = 4$$

$$y(x) = -1 + 4e^{x^3/3}$$

2. Find the general solution.

$$(\cos(x)y + x) + (\sin(x) + y^2)y' = 0$$

Leave your solution in implicit form. non-linear, first order, not separable

Try method of exact equations:

$M(x,y) + N(x,y)y' = 0$

Is it exact?

$M_y = \cos(x)$, $N_x = \cos(x) \Rightarrow M_y = N_x$ ✓ (exact)

Therefore, $F_x = M$ and $F_y = N$ for some function $F(x,y)$

Find $F(x,y)$

$F(x,y) = \int M\,dx = \int [\cos(x)y + x]\,dx = \sin(x)y + \frac{x^2}{2} + h(y)$

Match to $F_y = N$

$F_y = \sin(x) + h'(y)$. But $F_y = N = \sin(x) + y^2 \Rightarrow h'(y) = y^2$

Find $h(y)$

$h(y) = \int y^2\,dy = \frac{y^3}{3} + C$

Write $F(x,y)$

$F(x,y) = \sin(x)y + \frac{x^2}{2} + \frac{y^3}{3} + C$

3. Without solving the equation, determine where a unique solution is guaranteed to exist.

$$\ln(t)y'' - y' + \frac{1}{t-3}y = \sqrt{8-t}, \quad y(2) = 3.$$

2^{nd} order, linear, non-homogeneous

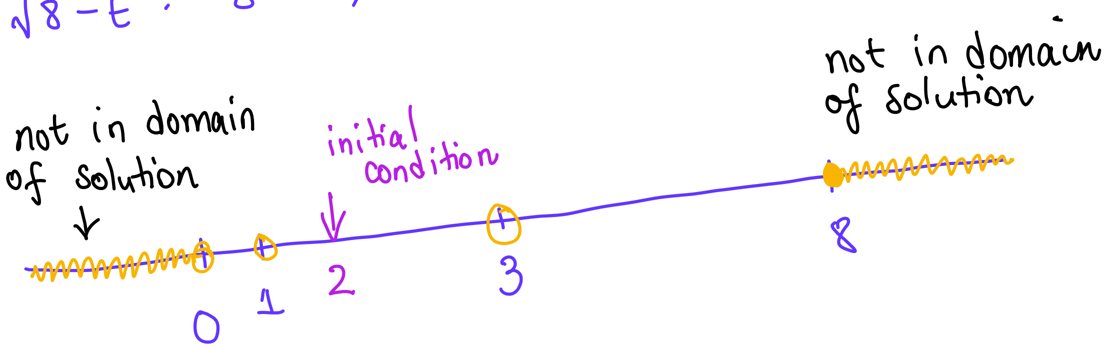
Standard form equation

$$y'' - \frac{1}{\ln(t)}y' + \frac{1}{\ln(t)(t-3)}y = \frac{1}{\ln(t)}\sqrt{8-t}$$

* $\frac{1}{\ln(t)}$: $t > 0$, $t \neq 1$ since $\ln(1) = 0$

* $\frac{1}{t-3}$: $t \neq 3$

* $\sqrt{8-t}$: $8 - t \geq 0 \Rightarrow 8 \geq t \Rightarrow t \leq 8$



Solution domain

$(1, 3)$

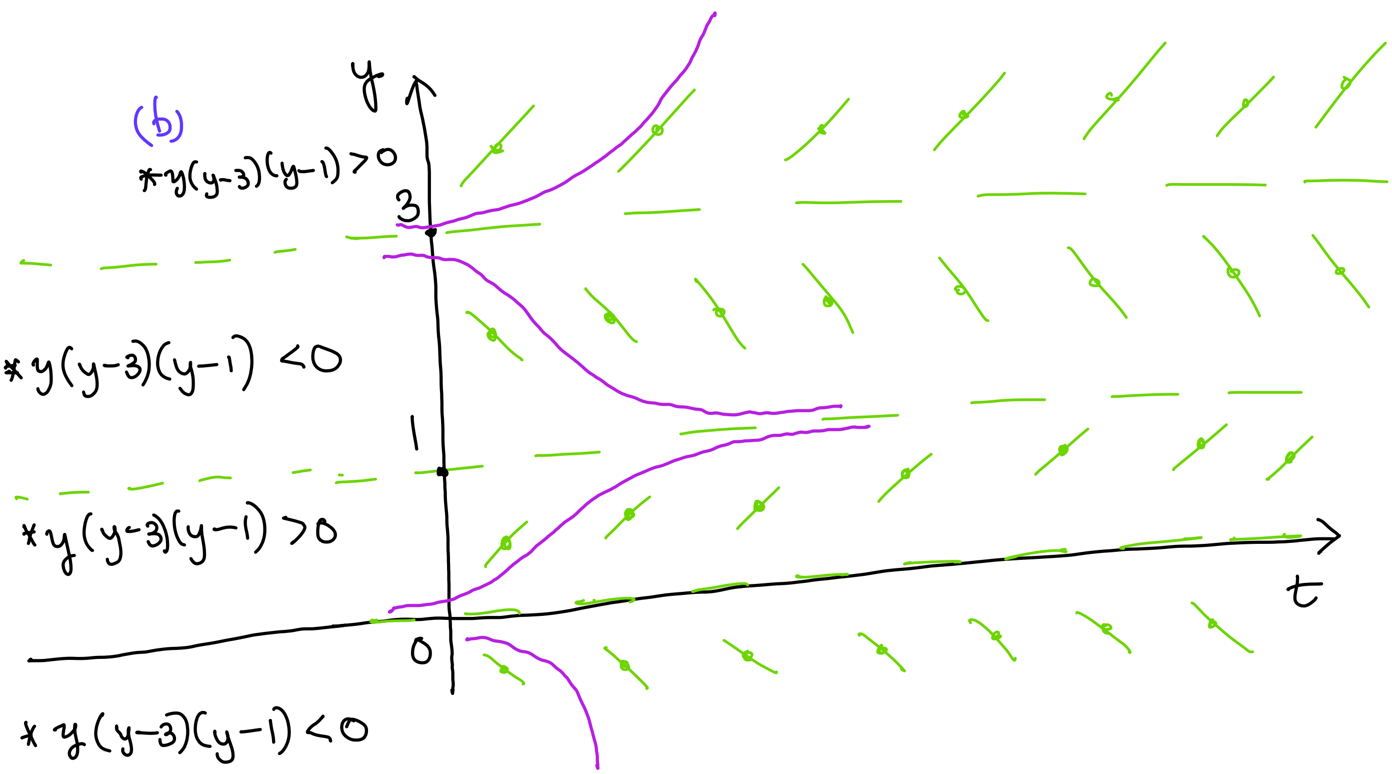
4. Consider the differential equation

$$y' = y^3 - 4y^2 + 3y.$$

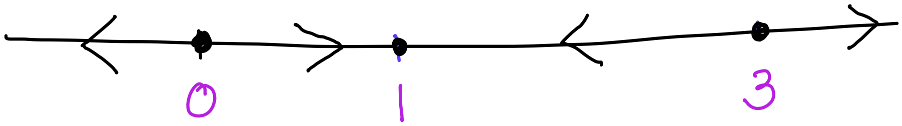
   (a) Find the equilibrium solutions.
   (b) Plot the direction field.
      - Draw a few example solutions on the direction field.
   (c) Draw the phase line diagram.
   (d) Determine the stability of each equilibrium point.

(a) Solve $y' = y^3 - 4y^2 + 3y = 0 \Rightarrow y(y^2 - 4y + 3) = 0$

$y(y-3)(y-1) = 0 \Rightarrow \boxed{y = 0, 1, 3}$



(c) Phase line



(d) Stability

$y = 0$ is unstable (source)

$y = 1$ is stable (sink)

$y = 3$ is unstable (source)

5. An object is initially at 100°C in a room with a constant temperature of 20°C. It cools according to Newton's law of cooling with a cooling constant $k = 0.05$ per minute. Find the time when the object's temperature reaches 50°C.

Newton's Law of Cooling

$u \to$ temperature of object

$T \to$ room temperature (20°C)

$\frac{du}{dt} = -k(u-T)$, $u(0) = 100°C$

$k = 0.05 = \frac{5}{100} = \frac{1}{20}$

$\Rightarrow \frac{du}{dt} = -k(u-20)$

$\Rightarrow \int \frac{1}{u-20}\,du = \int -k\,dt = -kt + C_1$

$\Rightarrow \ln|u-20| = -kt + C_1 \Rightarrow e^{\ln|u-20|} = e^{-kt+C_1} = e^{C_1} \cdot e^{-kt}$

$\Rightarrow |u-20| = C_2 e^{-kt}$, where $C_2 = e^{C_1}$

$\Rightarrow u - 20 = \pm C_2 e^{-kt} = C_3 e^{-kt}$ where $C_3 = \pm C_2$

$u = 20 + C_3 e^{-kt} \Rightarrow u(0) = 20 + C_3 = 100 \Rightarrow C_3 = 80$

$u(t) = 20 + 80 e^{-t/20}$

Find t when $u = 50$: $50 = 20 + 80 e^{-t/20}$

$\Rightarrow 30 = 80 e^{-t/20}$

$\Rightarrow \frac{30}{80} = e^{-t/20} \Rightarrow e^{-t/20} = 3/8 \Rightarrow \frac{-t}{20} = \ln(3/8)$

$t = -20\ln(3/8)$

$t = 20\ln(8/3)$

6. Find the general solution of the equation

$$u'' + 2u' + u = 0$$

2nd order, linear, constant coeff, homogeneous

Characteristic Equation

$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ (repeated)

Solution

$$u(t) = c_1 e^{-t} + c_2 t e^{-t}$$

Check:

fundamental set: $y_1 = e^{-t}$, $y_2 = te^{-t}$

$y_1' = -e^{-t}$, $y_1'' = e^{-t} \Rightarrow y_1'' + 2y_1' + y_1 = e^{-t} + 2(-e^{-t}) + e^{-t} = 0$ ✓

$y_2' = e^{-t} - te^{-t}$, $y_2'' = -e^{-t} - e^{-t} + te^{-t} \Rightarrow y_2'' + 2y_2' + y_2$

$$= (-2\cancel{e^{-t}} + \cancel{te^{-t}}) + 2(\cancel{e^{-t}} - \cancel{te^{-t}}) + \cancel{te^{-t}} = 0$$

Compute Wronskian

$$W(t) = y_1 y_2' - y_1' y_2 = e^{-t}(e^{-t} - te^{-t}) - (-e^{-t})(te^{-t})$$

$$= e^{-2t} - \cancel{te^{-2t}} + \cancel{te^{-2t}} = e^{-2t} \neq 0$$

solutions are independent

7. Find the general solution. Prove that it is indeed the general solution.

$$u'' + 6u' + 10u = 0$$

Characteristic eqn: $r^2+6r+10=0 \Rightarrow (r+3)^2+1=0$

$$(r+3)^2=-1$$

$$r+3=\pm i$$

$$r=-3\pm i$$

$$u(t)=c_1e^{-3t}\cos(t)+c_2e^{-3t}\sin(t)$$

Verify: $u_1=e^{-3t}\cos(t)$, $u_2=e^{-3t}\sin(t)$

$u_1'=-3e^{-3t}\cos(t)-e^{-3t}\sin(t)$, $u_1''=9e^{-3t}\cos(t)+3e^{-3t}\sin(t)$

$\qquad +3e^{-3t}\sin(t)-e^{-3t}\cos(t)$

$u_1''+6u_1'+10u_1=e^{-3t}\cos(t)\left[9-1\right]+e^{-3t}\sin(t)\left[3+3\right]$

$\qquad +6e^{-3t}\cos(t)\left[-3\right]+6e^{-3t}\sin(t)\left[-1\right]+10e^{-3t}\cos(t)$

$\qquad =e^{-3t}\cos(t)\left(\underbrace{8-18+10}_{\to 0}\right)+e^{-3t}\sin(t)\left(\underbrace{6-6}_{\to 0}\right)=0$ ✓

u_1 is a solution

$$u_2 = e^{-3t}\sin(t) \ , \quad u_2' = -3e^{-3t}\sin(t) + e^{-3t}\cos(t)$$

$$u_2'' = 9e^{-3t}\sin(t) - 3e^{-3t}\cos(t) - 3e^{-3t}\cos(t) - e^{-3t}\sin(t)$$

$$= 8e^{-3t}\sin(t) - 6e^{-3t}\cos(t)$$

$$u_2'' + 6u_2' + 10u_2 = e^{-3t}\cos(t)\left[-6\right] + e^{-3t}\sin(t)\,8$$

$$+6e^{-3t}\cos(t)\left[1\right] + 6e^{-3t}\sin(t)\left[-3\right]$$

$$+ 10e^{-3t}\sin(t)$$

$$= e^{-3t}\cos(t)\overset{\to 0}{\left[-6+6\right]} + e^{-3t}\sin(t)\overset{\to 0}{\left[8-18+10\right]}$$

$$= 0 \ \checkmark \qquad u_2 \text{ is a solution}$$

Compute Wronskian

$$W(t) = u_1 u_2' - u_1' u_2 = \left(e^{-3t}\cos(t)\right)\left(-3e^{-3t}\sin(t) + e^{-3t}\cos(t)\right)$$

$$- \left(-3e^{-3t}\cos(t) - e^{-3t}\sin(t)\right)\left(e^{-3t}\sin(t)\right)$$

$$= \cancel{-3e^{-6t}\cos(t)\sin(t)} + e^{-6t}\cos^2(t) + \cancel{3e^{-6t}\cos(t)\sin(t)}$$

$$+ e^{-6t}\sin^2(t)$$

$$= e^{-6t}\left(\overset{1}{\cancel{\cos^2(t) + \sin^2(t)}}\right) = e^{-6t} \neq 0$$

(independent solutions)

8. Find a particular solution to

2nd order, linear, non-homogeneous, non-constant coeffs

$$t^2y'' - 3ty' + 3y = 5t^2, \quad t > 0,$$

given that t and t^3 are solutions to the corresponding homogeneous equation.

use variation of parameters

$y_1(t) = t, \quad y_2(t) = t^3$

$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$

where $u_1(t) = \int \frac{-y_2(t)\,r(t)}{W(t)}\,dt$ standard form, $\quad u_2(t) = \int \frac{y_1(t)\,r(t)}{W(t)}\,dt$ standard form

Standard form: $y'' - \frac{3}{t}y' + \frac{3}{t^2} = 5, \quad t > 0$

$r(t) = 5$

$W(t) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 3t^3 - t^3 = 2t^3$

$u_1(t) = \int \frac{-t^3 \cdot 5}{2t^3}\,dt = -\frac{5}{2}t, \quad u_2 = \int \frac{t \cdot 5}{2t^3}\,dt = \frac{5}{2}\int \frac{1}{t^2}\,dt$

$= -\frac{5}{2}t^{-1}$

$y_p(t) = -\frac{5}{2}t^2 - \frac{5}{2}t^2 = -5t^2$ particular solution

$y(t) = c_1 t + c_2 t^3 - 5t^2$ general solution

9. Find a particular solution.

use undetermined coefficients

$$y'' - 3y' + 2y = 4e^t + 5$$

Homogeneous solution

$r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow r = 1, 2$

$y_c(t) = c_1 e^t + c_2 e^{2t}$

Particular solution : Note $4e^t$ is a homogeneous solution

Choose : $y_p^{(1)}(t) = Ate^t$, $y_p^{(2)}(t) = B$

$y_p^{(1)\prime} = Ae^t + Ate^t$, $y_p^{(1)\prime\prime} = Ae^t + Ae^t + Ate^t = 2Ae^t + Ate^t$

$y_p^{(1)\prime\prime} - 3y_p^{(1)\prime} + 2y_p^{(1)} = 2Ae^t + \cancel{Ate^t} - 3Ae^t - \cancel{3Ate^t} + \cancel{2Ate^t}$
$= -Ae^t = 4e^t \Rightarrow A = -4$

$y_p^{(2)} = B$, $y_p^{(2)\prime} = 0$, $y_p^{(2)\prime\prime} = 0$:

$y_p^{(2)\prime\prime} - 3y_p^{(2)\prime} + 2y_p^{(2)} = 2B = 5 \Rightarrow B = 5/2$

$y_p(t) = y_p^{(1)}(t) + y_p^{(2)}(t) = -4te^t + \frac{5}{2}$

10. Suppose there is a 30 N mass hanging on a spring. When the mass was attached to the spring, the spring stretched by 40 cm. When the mass is moving 5 m/s, it experiences a damping force of 15 N. There is an external upward (negative) force of 10 N acting on the mass for the first 10 seconds, after which there is no external force. Initially the mass is sent into motion with a downward (positive) velocity of 50 cm/s from the equilibrium position. (Use $g = 10\,\text{m/s}^2$.)

(a) Write down an initial value problem that describes the motion of the mass.

$$mu'' + cu' + ku = F(t)$$

$u(0) = 0$ m (equilibrium)

$u'(0) = 0.5$ m/s

$$m = \frac{\text{Weight}}{g} = \frac{30\text{N}}{10\text{m/s}^2} = 3\text{ kg}$$

$$k = \frac{\text{weight}}{\text{extension}} = \frac{30\text{N}}{0.4\text{m}} = 75\text{ N/m}$$

$$c = \frac{\text{damping force}}{\text{velocity}} = \frac{15\text{N}}{5\text{m/s}} = 3\text{ Ns/m}$$

$$F(t) = -10(u_0 - u_{10})\text{ N} = -10 + 10u_{10}(t)$$

$$\boxed{\begin{gathered}3u'' + 3u' + 75u = -10 + 10u_{10}(t)\\ u(0) = 0,\ u'(0) = 0.5\end{gathered}}$$

(b) Is the system over, under, or critically damped?

$$3r^2 + 3r + 75 = 0 \Rightarrow r = \frac{-3 \pm \sqrt{3^2 - 4\cdot 3\cdot 75}}{2\cdot 3}$$

negative discriminant

$$= -\frac{1}{2} \pm \sqrt{891}\, i$$

underdamped

11. Solve the initial value problem. Laplace transforms

$$y'' + y = u_1(t), \quad y(0) = 0, \quad y'(0) = 0.$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} = s^2 Y(s)$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{u_1(t)\} = \frac{e^{-s}}{s}$$

$$(s^2+1)Y(s) = \frac{e^{-s}}{s} \Rightarrow Y(s) = \frac{e^{-s}}{s(s^2+1)} = e^{-s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 1 = A(s^2+1) + (Bs+C)s$$

$s=0$: $A = 1$, $s=1$: $1 = 2+B+C$

$\Rightarrow B+C = -1$

$s=-1$: $1 = 2+B-C \Rightarrow B-C = -1$

$B = -1$, $C = 0$

$$\boxed{y(t) = u_1(t)\left[1 - \cos(t-1)\right]}$$

12. Using the **definition** of the Laplace transform, show that $\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2\,dt = -\frac{t^2}{s}e^{-st}\Big|_0^\infty - \frac{2t}{s^2}e^{-st}\Big|_0^\infty - \frac{2}{s^3}e^{-st}\Big|_0^\infty$$

By parts:

t^2	$+$	e^{-st}
$2t$	$-$	$-\frac{1}{s}e^{-st}$
2	$+$	$\frac{1}{s^2}e^{-st}$
0		$-\frac{1}{s^3}e^{-st}$

$$= \frac{2}{s^3} \quad (\text{if } s>0)$$

13. Find the general solution in the form of a power series centered at $x = 0$.

$$y'' + xy = 0. \qquad y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \xrightarrow{\text{shift}} y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$xy = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \xrightarrow{\text{shift}} \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$y'' + xy = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_{n-1}\right] x^n = 0$$

$2a_2 = 0 \Rightarrow a_2 = 0$: Recurrence relation:

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0 \Rightarrow a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)}$$

for $n \geq 1$ only

Find 2 independent solutions

Choose: $y_1(0) = 1$, $y_1'(0) = 0 \Rightarrow a_0 = 1$, $a_1 = 0$

$a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = \frac{-a_0}{6} = \frac{-1}{6}$, $a_4 = 0$, $a_5 = 0$, $a_6 = \frac{-a_3}{30} = \frac{1}{180}$

$$y_1(t) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots$$

Choose: $y_2(0) = 0$, $y_2'(0) = 1 \Rightarrow a_0 = 0$, $a_1 = 1$

$a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = 0$, $a_4 = \frac{-a_1}{12} = \frac{-1}{12}$, $a_5 = 0$, $a_6 = 0$, $a_7 = \frac{-a_4}{42} = \frac{1}{504}$

$$y_2(t) = t - \frac{1}{12}t^4 + \frac{1}{504}t^7 + \cdots$$

General solution: $\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t)}$

14. Find the general solution to the system of differential equations.

$$x_1' = 3x_1 + x_2$$
$$x_2' = -x_1 + x_2$$

write in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvalues

$$\lambda^2 - tr(A)\lambda + det(A) = 0$$

$$\lambda^2 - 4\lambda + 4 = 0, \quad \lambda = 2 \text{ (repeated)}$$

Eigenvectors

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a+b=0 \\ a=-b \end{matrix} \Rightarrow v = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$w_1(t) = e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad \text{Set } w_2(t) = e^{2t}\left(t\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right)$$

where $(A - \lambda I)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{v} \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$u_1 + u_2 = -1 \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

General solution:

$$X(t) = c_1 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$