3.7–3.8 MECHANICAL VIBRATIONS

Review

- Standard equation for a **mass and spring system**

$$mu'' + \gamma u' + ku = F(t)$$

- Standard equation for **electronic circuits**

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

- Damping
  - Underdamped
  - Critically damped
  - Overdamped

- To find the amplitude of a solution, use the following formula.

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta),$$

  where $R = \sqrt{A^2 + B^2}$ and $\delta = \arctan(B/A)$. The **amplitude** is R, and the **angular frequency** is ω.

- The **period** is $2\pi/$(angular frequency).

- **Natural frequency:** The frequency of the system when there is no damping and no external force.

- **Resonance** occurs when there is no damping and the external force of a system matches the natural frequency of the system. When this happens, the oscillations grow unboundedly large.

Exercise 1

A mass of 3 kg is attached to a spring. When the mass is attached, the spring stretches an additional 50 cm. There is no damping. The mass is initially stretched an additional 10 cm and pushed with an initial downward velocity of 2 m/s. Find the position of the mass over time. Find the frequency, period, and amplitude. (Use $g = 10\,\mathrm{m/s^2}$.)

Exercise 2

A spring has a mass attached to it with weight 5 N. When the spring is attached, the spring stretches 200 cm. When the mass is moving 2 m/s, there is a damping force of 4 N. We start the spring from the equilibrium position with an initial upward velocity of 3 m/s. Find the location of the mass over time. (Use $g = 10\,\mathrm{m/s^2}$.)

Exercise 3

A spring has a mass of 1 kg attached to it with a spring constant of 4 N/m. There is an external force $\sin(2t)$ pushing on the mass. Suppose the mass starts from equilibrium at rest. Find the location of the mass over time. What happens as $t \to \infty$?

Exercise 4

Consider the equation

$$3u'' + \gamma u' + 10u = 0.$$

Find the value of γ that makes the system critically damped.

6.1: DEFINITION OF LAPLACE TRANSFORM

Review

- The Laplace transform is defined by $\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$.
- For many functions, you can just look up the Laplace transform in the table.

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n $(n = 1, 2, ...)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n$ $(n = 1, 2, ...)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

- The Laplace transform is also linear: $\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}$.
- To take the **inverse Laplace transform**, you can also use the table. However, if your function does not match the things in the table, then you need to first do partial fractions.
- Partial fractions review
  - Simple roots
  - Irreducible quadratics
  - Repeated roots

Exercise 5

Using the definition of the Laplace transform, compute the Laplace transform of $3t$.

Exercise 6

Find the Laplace transform of $\begin{cases} 3 & 0 \le t \le 5, \\ e^{2t} & t > 5. \end{cases}$

Exercise 7

Find the Laplace transform of $g(t) = 6t^4 - 3\cos(2t) + e^{4t}\sin(t)$.

Exercise 8

Find the inverse transform of $F(s) = \dfrac{2}{s^4}$.

Exercise 9

Find the inverse transform of $G(s) = \dfrac{3}{s^2 + 3}$.

Exercise 10

Find the inverse transform of $Y(s) = \dfrac{2}{s^2 + 4s + 7}$.

Exercise 11

Find the inverse transform of $F(s) = \dfrac{s+1}{(s-2)(s^2+2s+3)}$.

Exercise 12

Find the inverse transform of $Y(s) = \dfrac{s}{(s-1)^2(s+2)}$.