Math 151 - Week-In-Review 11

Topics for the week:

4.4 Indeterminate Forms and L'Hopital's Rule
4.7 Optimization

4.4 Indeterminate Forms and L'Hopital's Rule

1. Evaluate the following limit: $\lim_{x\to 0} \dfrac{e^x - \cos x}{x^2+3x}$

$$\lim_{x\to 0}\left[\frac{e^x - \cos(x)}{x^2+3x}\right] = \lim_{x\to 0}\left[\frac{\overbrace{e^x - \cos(x)}^{\to 0}}{\underbrace{x^2+3x}_{\to 0}}\right]$$

$$\overset{L.H.}{=} \lim_{x\to 0}\left[\frac{e^x - (-\sin(x))}{2x+3}\right]$$

$$= \lim_{x\to 0}\left[\frac{\overbrace{e^x + \sin(x)}^{\to 1}}{\underbrace{2x+3}_{\to 3}}\right]$$

$$\lim_{x\to 0}\left[\frac{e^x - \cos(x)}{x^2+3x}\right] = \frac{1}{3}$$

2. Evaluate $\lim_{x\to \frac{\pi}{2}^-} (\sec(x) - \tan(x))$

$$\lim_{x\to \frac{\pi}{2}^-}\left[\underbrace{\sec(x)}_{\to\infty} - \underbrace{\tan(x)}_{\to\infty}\right] = \lim_{x\to \frac{\pi}{2}^-}\left[\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right]$$

$$= \lim_{x\to \frac{\pi}{2}^-}\left[\frac{\overbrace{1-\sin(x)}^{\to 0}}{\underbrace{\cos(x)}_{\to 0}}\right]$$

$$\overset{L.H.}{=} \lim_{x\to \frac{\pi}{2}^-}\left[\frac{\overbrace{-\cos(x)}^{\to 0}}{\underbrace{-\sin(x)}_{\to -1}}\right]$$

$$\lim_{x\to \frac{\pi}{2}^-}\left[\sec(x) - \tan(x)\right] = 0$$

3. Evaluate $\lim_{x \to 0^-} \left(\frac{x - \sin(x)}{x^3} \right)$

$$\lim_{x \to 0^-} \left[\frac{\overset{\to 0}{x} - \overset{\to 0}{\sin(x)}}{\underset{\to 0}{x^3}} \right] = \lim_{x \to 0^-} \left[\frac{\overbrace{x - \sin(x)}^{\to 0}}{\underbrace{x^3}_{\to 0}} \right]$$

$$\overset{\text{L.H.}}{=} \lim_{x \to 0^-} \left[\frac{1 - \overset{\to 1}{\cos(x)}}{3x^2} \right]$$

$$= \lim_{x \to 0^-} \left[\frac{\overbrace{1 - \cos(x)}^{\to 0}}{\underbrace{3x^2}_{\to 0}} \right]$$

$$\overset{\text{L.H}}{=} \lim_{x \to 0^-} \left[\frac{\overbrace{\sin(x)}^{\to 0}}{\underbrace{6x}_{\to 0}} \right]$$

$$\overset{\text{L.H.}}{=} \lim_{x \to 0^-} \left[\frac{\cos(x)}{6} \right]$$

$$\lim_{x \to 0^-} \left[\frac{x - \sin(x)}{x^3} \right] = \frac{1}{6}$$

4. Evaluate $\lim_{x \to 0^+} (x \cot(x))$.

$$\lim_{x \to 0^+} \left[\underset{\to 0}{x} \, \underset{\to \infty}{\cot(x)} \right] = \lim_{x \to 0^+} \left[\frac{\overbrace{x}^{\to 0}}{\underbrace{\tan(x)}_{\to 0}} \right]$$

$$\overset{\text{L.H.}}{=} \lim_{x \to 0^+} \left[\frac{1}{\sec^2(x)} \right]$$

$$= \lim_{x \to 0^+} \left[\cos^2(x) \right]$$

$$\lim_{x \to 0^+} \left[x \cot(x) \right] = 1$$

5. Evaluate $\lim_{x\to\infty}\left(\frac{x^2}{2^x}\right)$.

$$\lim_{x\to\infty}\left[\frac{\overset{\to\infty}{x^2}}{\underset{\to\infty}{2^x}}\right] \overset{\text{L.H.}}{=} \lim_{x\to\infty}\left[\frac{\overbrace{2x}^{\to\infty}}{\underbrace{2^x\ln(2)}_{\to\infty}}\right]$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty}\left[\frac{\overbrace{2}^{\to 2}}{\underbrace{2^x(\ln(2))^2}_{\to\infty}}\right]$$

$$\lim_{x\to\infty}\left[\frac{x^2}{2^x}\right] = 0$$

6. Evaluate $\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)$

$$\lim_{x\to\infty}\left[\underset{\to\infty}{x} - \underset{\to\infty}{\sqrt{x^2+1}}\right] = \lim_{x\to\infty}\left[x - \sqrt{x^2\left(1+\frac{1}{x^2}\right)}\right]$$

Note: $\sqrt{x^2} = x$ for $x\to\infty$

$$= \lim_{x\to\infty}\left[x - x\sqrt{1+1/x^2}\right]$$

$$= \lim_{x\to\infty}\left[\underset{\to\infty}{x}\left(1-\underbrace{\sqrt{1+1/x^2}}_{\to 0}\right)\right]$$

$$= \lim_{x\to\infty}\left[\frac{\overbrace{1-(1+\overset{\to 0}{1/x^2})^{1/2}}}{\underbrace{1/x}_{\to 0}}\right]$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty}\left[\frac{-\frac{1}{2}(1+1/x^2)^{-1/2}\cdot -2/x^3}{-1/x^2}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\overbrace{-1}^{\to -1}}{\underbrace{(1+1/x^2)^{1/2}\cdot x}_{\to\infty}}\right]$$

$$\lim_{x\to\infty}\left[x-\sqrt{x^2+1}\right] = 0$$

7. Evaluate $\lim_{x\to\left(\frac{\pi}{4}\right)^+}\left((\tan(x))^{\tan(2x)}\right)$.

$$\lim_{x\to\left(\frac{\pi}{4}\right)^+}\left[\underbrace{(\tan(x))}_{\to 1}{}^{\overbrace{\tan(2x)}^{\to -\infty}}\right]$$

Let $y=(\tan(x))^{\tan(2x)}$ then

$$\ln(y)=\tan(2x)\cdot\ln(\tan(x))$$

$$\lim_{x\to\frac{\pi}{4}^+}(\ln(y))=\lim_{x\to\frac{\pi}{4}^+}\left[\underbrace{\tan(2x)}_{\to -\infty}\cdot\underbrace{\ln(\tan(x))}_{\to 0}\right]_{\to 0}$$

$$\lim_{x\to\frac{\pi}{4}^+}(\ln(y))=\lim_{x\to\frac{\pi}{4}^+}\left[\frac{\overbrace{\ln(\tan(x))}^{\to 0}}{\underbrace{\cot(2x)}_{\to 0}}\right]$$

$$\lim_{x\to\frac{\pi}{4}^+}(\ln(y))\overset{L.H.}{=}\lim_{x\to\frac{\pi}{4}^+}\left[\frac{\overbrace{\frac{1}{\tan(x)}\cdot\sec^2(x)}^{\to 2}}{\underbrace{-2\csc^2(2x)}_{\to -2}}\right]$$

$$\lim_{x\to\frac{\pi}{4}^+}(y)=e^{-1}$$

8. Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x}$.

$$\lim_{x\to\infty}\left[\underbrace{\left(1+\frac{1}{2x}\right)}_{\to 1}{}^{\overbrace{3x}^{\to\infty}}\right]$$

Let $y=\left(1+\frac{1}{2x}\right)^{3x}$ then $\ln(y)=3x\ln\left(1+\frac{1}{2x}\right)$

$$\lim_{x\to\infty}(\ln(y))=\lim_{x\to\infty}\left[\underbrace{3x}_{\to\infty}\,\underbrace{\ln(1+1/2x)}_{\to 0}\right]_{\to 0}$$

$$\lim_{x\to\infty}(\ln(y))=\lim_{x\to\infty}\left[\frac{\overbrace{3\ln(1+1/2x)}^{\to 0}}{\underbrace{1/x}_{\to 0}}\right]$$

$$\lim_{x\to\infty}(\ln(y))\overset{L.H.}{=}\lim_{x\to\infty}\left[\frac{\frac{3}{1+\frac{1}{2x}}\cdot\frac{-1}{2x^2}}{-1/x^2}\right]$$

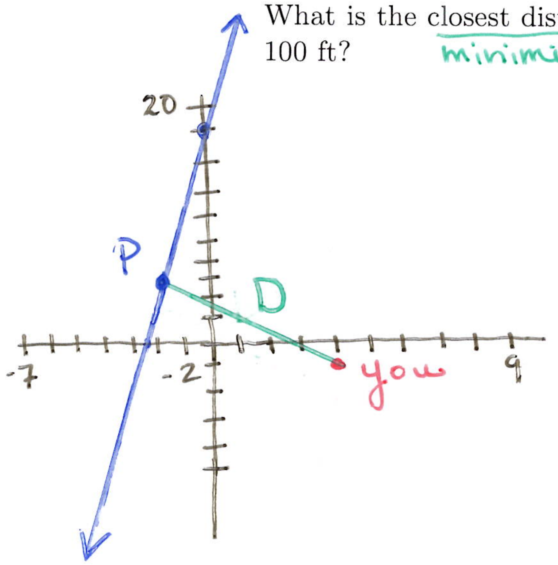
$$\lim_{x\to\infty}(\ln(y))=\lim_{x\to\infty}\left[\frac{\overbrace{3}^{\to 3}}{\underbrace{2+1/x}_{\to 2}}\right]$$

$$\lim_{x\to\infty}(y)=e^{3/2}$$

4.7 Optimization

9. The state is building a new highway that you pass near the town you live in. On the map your house is located at the point $(4, -2)$ and the new highway is to follow the line $y = 8x + 18$. What is the closest distance the highway will be from your house, if one unit on the map is 100 ft?

minimize



Highway
$y = 8x + 18 \qquad x \in (-\infty, \infty)$

$P(x_2, y_2) = (x, 8x+18)$

$(x_1, y_1) = (4, -2)$

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D = \sqrt{(x-4)^2 + (8x + 18 - (-2))^2}$$

Minimize: $D(x) = \sqrt{(x-4)^2 + (8x+20)^2}$

$$D(x) = \sqrt{x^2 - 8x + 16 + 64x^2 + 320x + 400}$$

$$D(x) = \sqrt{65x^2 + 312x + 416}$$

$$D'(x) = \frac{1}{2}(65x^2 + 312x + 416)^{-1/2}(130x + 312)$$

$$D'(x) = \frac{130x + 312}{2(65x^2 + 312x + 416)^{1/2}}$$

Critical Numbers:

$$\frac{130x + 312}{2\sqrt{65x^2 + 312x + 416}} = 0$$

$$130x + 312 = 0$$

$$x = \frac{-12}{5}$$

behavior ↘ min ↗

sign − +

$-\frac{12}{5}$

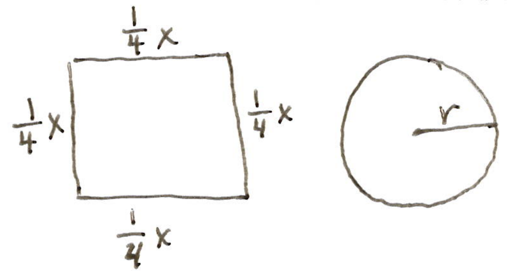
$$D\left(\tfrac{-12}{5}\right) = \sqrt{\left(\tfrac{-12}{5} - 4\right)^2 + \left(8\left(\tfrac{-12}{5}\right) + 20\right)^2}$$

$$= \sqrt{\left(\tfrac{-32}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2}$$

$$= \sqrt{\frac{1040}{25}} \text{ units}$$

≈ 6.45 unit $\cdot$ 100 ft $= 645$ ft

10. A piece of wire 12 inches long is being used to make up to two figures: an square and a circle. How should the wire be divided so that the total area enclosed is a maximum? A minimum?



Let x = the perimeter of the square

12in

x

So Circumference is

$2\pi r = 12 - x$

$r = \dfrac{12-x}{2\pi}$ $\quad x \in [0,12]$

$A = \left(\frac{1}{4}x\right)^2 + \pi r^2$

$A(x) = \frac{1}{16}x^2 + \pi\left(\frac{12-x}{2\pi}\right)^2$

$A'(x) = \frac{1}{8}x + 2\pi\left(\frac{12-x}{2\pi}\right)\cdot\left(\frac{-1}{2\pi}\right)$

$A'(x) = \frac{1}{8}x - \left(\frac{12-x}{2\pi}\right)$

Critical Numbers:

$0 = \frac{1}{8}x - \left(\frac{12-x}{2\pi}\right)$

$\frac{12-x}{2\pi} = \frac{x}{8}$

$96 - 8x = 2\pi x$

$96 = 2\pi x + 8x$

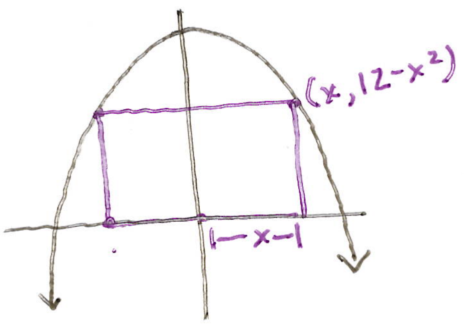
$\frac{96}{2\pi + 8} = x \approx 6.72 \text{ in}$

x	$A(x)$
0	$\frac{1}{16}(0)^2 + \pi\left(\frac{12-(0)}{2\pi}\right)^2 = \frac{36}{\pi} \approx 11.5$
$\frac{96}{2\pi+8}$	$\frac{1}{16}\left(\frac{96}{2\pi+8}\right)^2 + \pi\left(\frac{12-\frac{96}{2\pi+8}}{2\pi}\right)^2 \approx 5.04$
12	$\frac{1}{16}(12)^2 + \pi\left(\frac{12-(12)}{2\pi}\right)^2 = 9$

The maximum area is when you do not cut the wire and use it all in the circle.

The minimum area is when you cut the wire and use $\frac{96}{2\pi+8}$ in for the square.

11. What are the dimensions of the largest rectangle that can be inscribed in the area bounded by the curve $y = 12 - x^2$ and the x-axis?



$A = 2x\,y$

$A = 2x(12 - x^2)$

Maximize: $A(x) = 24x - 2x^3$

$\frac{dA(x)}{dx} = 24 - 6x^2$

$y = 12 - x^2 \qquad x \in [0, \sqrt{12}]$

$12 = x^2$

$\sqrt{12} = |x|$

Critical Number:

$0 = 24 - 6x^2$

$6x^2 = 24$

$x^2 = 4$

$|x| = 2 \qquad x = -2$ is not in the domain

$x = 2$

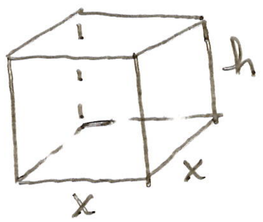
x	$A(x)$
0	$2(0)(12-(0)^2) = 0$
2	$2(2)(12-(2)^2) = 32$
$\sqrt{12}$	$2(\sqrt{12})(12-(\sqrt{12})^2) = 0$

$x = 2 \Rightarrow 2x = 4$ and $y = 8$

The rectangle should be 4×8 rectangle.

12. A box with a square base and open top must have a volume of 64 m^3. Determine the dimensions of the box that minimized the amount of material used.

Surface area



$$A = x^2 + 4xh$$

$$A = x^2 + 4x\left(\frac{64}{x^2}\right)$$

Minimize: $A = x^2 + \frac{256}{x}$

$$V = 64\,m^3 = x^2 h$$

$$\frac{64}{x^2} = h \qquad x \in (0, \infty)$$

$$\frac{dA}{dx} = 2x - \frac{256}{x^2}$$

Critical Numbers:

$$0 = 2x - \frac{256}{x^2}$$

$$0 = \frac{2x^3 - 256}{x^2}$$

$2x^3 - 256 = 0$ or $x^2 = 0$

$x^3 = 128$ $\quad$ $x = 0$ not in domain

$x = \sqrt[3]{128}$

$$\frac{d^2A}{dx^2} = 2 + \frac{512}{x^3}$$

$$\left.\frac{d^2A}{dx^2}\right|_{x=\sqrt[3]{128}} = 2 + \frac{512}{(\sqrt[3]{128})^3} = 2 + \frac{512}{128}$$

is +

A is ∪

$x = \sqrt[3]{128}$ is where the minimum is

The dimensions of the box are $\sqrt[3]{128}$ m $\times$ $\sqrt[3]{128}$ m $\times$ $\frac{64}{(\sqrt[3]{128})^2}$ m

13. A family of functions, $s(t) = At^pe^{-kt}$, are called surge functions because the function has a "surge" prior to decaying slowly. One example of this is the concentration of medication in the blood system after an injection. If we assume $A = 4$, $p = 1$, and $k = 1$, determine the time, t, in which $s(t)$ is maximized.

$S(t) = 4te^{-t} \qquad t \in [0, \infty)$

$s'(t) = 4e^{-t} + (-e^{-t})(4t)$

$s'(t) = 4e^{-t}(1-t)$

Critical Numbers:

$0 = 4e^{-t}(1-t)$

$4e^{-t} \neq 0$ or $1-t=0$

$1 = t$

$s''(t) = -4e^{-t}(1-t) + (-1)(4e^{-t})$

$= -4e^{-t}(1-t+1)$

$s''(t) = -4e^{-t}(2-t)$

$s''(1) = -4e^{-(1)}(2-1)$

is negative

S(t) is $\cap$

The time is 1 unit.

