$$\frac{1}{x-a} \longrightarrow \frac{A}{x-a}\ ; \quad \frac{1}{(x-a)^n} \longrightarrow \frac{A}{x-a} + \frac{B}{(x-a)^2} + \dots + \frac{C}{(x-a)^n}\ ; \quad \underset{(b^2-4ac<0)}{\frac{1}{ax^2+bx+c}} \longrightarrow \frac{Ax+B}{ax^2+bx+c}$$

Math 152/172 **WEEK in REVIEW 5** **Spring 2025.**

1. Write the correct partial fraction expansion. Do not solve for unknown constants.

(a) $\dfrac{x^{\textcircled{5}} + +2x^4 - x^3 + 3x^2 + x - 1}{x^3 + x^{\textcircled{4}}}$ — improper

$$= x+1+\frac{-2x^3+3x^2+x-1}{x^3+x^4} = x+1+\frac{-2x^3+3x^2+x-1}{\underbrace{x^3}_{\text{linear repeated}}\underbrace{(x+1)}_{\text{linear nonrepeated}}}$$

$$= \boxed{x+1+\frac{A}{x+1}+\frac{B}{x}+\frac{C}{x^2}+\frac{D}{x^3}}$$

$$\begin{array}{r|l} & x+1 \\ \hline x^4+x^3 & x^5+2x^4-x^3+3x^2+x-1 \\ & -\ x^5+x^4 \\ \hline & x^4-x^3 \\ & x^4+x^3 \\ \hline & -2x^3+3x^2+x-1 \end{array}$$

(b) $\dfrac{x}{x^2+x-6} = \dfrac{x}{(x+3)(x-2)} = \boxed{\dfrac{A}{x+3} + \dfrac{B}{x-2}}$

$x^2+x-6 = (x+3)(x-2)$

(c) $\dfrac{1+16x}{\underbrace{(2x-3)}_{\text{linear nonrepeated}}\underbrace{(x^2+4)}_{\text{quadratic irreducible}}\underbrace{(x+5)^2}_{\text{linear repeated}}} = \boxed{\dfrac{A}{2x-3} + \dfrac{Bx+C}{x^2+4} + \dfrac{D}{x+5} + \dfrac{E}{(x+5)^2}}$

1

2. Evaluate the integral.

(a) $\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

partial fractions: $\frac{5x+1}{(2x+1)(x-1)} = \frac{\cancel{A}^{1}}{2x+1} + \frac{\cancel{B}^{2}}{x-1}$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A(x-1)+B(2x+1)}{(2x+1)(x-1)}$$

$$5x+1 = A(x-1)+B(2x+1)$$

$x=1$: $5(1)+1 = A(1-1)+B(2+1) \longrightarrow 6=3B \longrightarrow \boxed{B=2}$

$x=-1/2$: $5(-\frac{1}{2})+1 = A(-\frac{1}{2}-1)+B(-1+1) \longrightarrow -\frac{3}{2} = -\frac{3}{2}A \longrightarrow \boxed{A=1}$

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int\left[\frac{1}{2x+1}+\frac{2}{x-1}\right]dx = \int\frac{dx}{2x+1} + 2\int\frac{dx}{x-1}$$

$u=2x+1$

$du=2dx$

$dx=\frac{du}{2}$

$$= \int\frac{du}{2u} + 2\ln|x-1| = \frac{1}{2}\ln|u| + 2\ln|x-1| + C$$

$$= \boxed{\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C}$$

(b) $\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$

Partial fractions.

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{\cancel{A}^{-2}}{x+1} + \frac{\cancel{B}^{1}}{(x+1)^2} + \frac{\cancel{C}^{3}}{x+2}$$

linear repeated

$$\frac{x^2+x+1}{(x+1)^2\,(x+2)} = \frac{A(x+1)(x+2)+B(x+2)+C(x+1)^2}{(x+1)^2\,(x+2)}$$

$$x^2+x+1 = A(x+1)(x+2)+B(x+2)+C(x+1)^2$$

$x=-1$ $\quad (-1)^2-1+1 = A(-1+1)(-1+2)+B(-1+2)+C(-1+1)^2$

$$\boxed{1=B}$$

$x=-2$ $\quad (-2)^2-2+1 = A(-2+1)(-2+2)+B(-2+2)+C(-2+1)^2$

$$\boxed{3=C}$$

$x=0$ $\quad 0+0+1 = A(1)(2)+B(2)+C(1)^2$

$$1 = 2A+2B+C$$

$$A = \frac{1}{2}(1-2\cancel{B}^{1}-\cancel{C}^{3}) = -\frac{4}{2} = -2 \longrightarrow \boxed{A=-2}$$

$$\int \frac{x^2+x+1}{(x+1)^2\,(x+2)}\,dx = \int\left[-\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}\right]dx$$

$$= -2\ln|x+1| + \frac{(x+1)^{-2+1}}{-2+1} + 3\ln|x+2| + C$$

$$\boxed{= -2\ln|x+1| - \frac{1}{x+1} + 3\ln|x+2| + C}$$

(c) $\int \frac{x^2\,dx}{x^4-81}$

$$x^4-81 = (x^2)^2 - 9^2 = (x^2-9)(x^2+9)$$

$$= \underbrace{(x-3)}_{\text{linear}}\underbrace{(x+3)}_{\text{linear}}\underbrace{(x^2+9)}_{\text{quadratic irreducible.}}$$

$$\frac{x^2}{x^4-81} = \frac{x^2}{(x-3)(x+3)(x^2+9)} = \frac{\overset{1/12}{\cancel{A}}}{x-3} + \frac{\overset{-1/12}{\cancel{B}}}{x+3} + \frac{\overset{0}{\cancel{C}}x+\overset{1/2}{\cancel{D}}}{x^2+9}$$

$$\frac{x^2}{(x-3)(x+3)(x^2+9)} = \frac{A(x+3)(x^2+9)+B(x-3)(x^2+9) + (Cx+D)\overbrace{(x-3)(x+3)}^{(x^2-9)}}{(x-3)(x+3)(x^2+9)}$$

$$x^2 = A(x+3)(x^2+9)+B(x-3)(x^2+9)+(Cx+D)(x^2-9)$$

$\underline{x=3}$

$$9 = A(3+3)(9+9)+B(3-3)(9+9)+(3C+D)(9-9)$$

$$\frac{9}{9} = \frac{6(18)A}{9} \rightarrow 1 = 12A \longrightarrow \boxed{A = \frac{1}{12}}$$

$\underline{x=-3.}$

$$9 = A(-3+3)(9+9)+B(-3-3)(9+9)+(-3C+D)(9-9)$$

$$9 = -6(18)B \longrightarrow \boxed{B = -\frac{1}{12}}$$

$\underline{x=0}$

$$\frac{0}{9} = \frac{27A-27B-9D}{9} \rightarrow D = 3A-3B = \frac{3}{12}+\frac{3}{12} = \frac{6}{12}$$

$$\boxed{D = \frac{1}{2}}$$

$\underline{x=1}$

$$1 = 40A-20B+(C+D)(-8)$$

$$1 = 40A-20B-8C-8D$$

$$8C = 40A-20B-8D-1$$

$$8C = \frac{40}{12}+\frac{20}{12}-\frac{8}{2}-1$$

$$8C = 5-5 = 0 \longrightarrow \boxed{C=0}$$

$$\int \frac{x^2dx}{x^4-81} = \int \left[\frac{1}{12}\cdot\frac{1}{x-3} - \frac{1}{12}\frac{1}{x+3} + \frac{1}{2}\frac{1}{x^2+9}\right]dx$$

$$= \boxed{\frac{1}{12}\ln|x-3| - \frac{1}{12}\ln|x+3| + \frac{1}{2}\cdot\frac{1}{3}\arctan\frac{x}{3} + C}$$

(d) $\int \frac{x^5 - x^4 - 2x^2 + 2x + 5}{x^4 + x^3}\,dx$

separate the whole part.

$$\begin{array}{r|l} & x-2 \\ \hline x^4+x^3 & x^5 - x^4 - 2x^2 + 2x+5 \\ & \underline{x^5 + x^4} \\ & -2x^4 - 2x^2 \\ & \underline{-2x^4 - 2x^3} \\ & \boxed{2x^3 - 2x^2 + 2x + 5} \end{array}$$

$$\frac{x^5-x^4-2x^2+2x+5}{x^4+x^3} = x-2 + \frac{2x^3-2x^2+2x+5}{\boxed{x^3}\boxed{(x+1)}}$$

linear

linear repeated

Partial fractions.

$$\frac{2x^3-2x^2+2x+5}{x^3(x+1)} = \frac{\overset{1}{\cancel{A}}}{x} + \frac{\overset{-3}{\cancel{B}}}{x^2} + \frac{\overset{5}{\cancel{C}}}{x^3} + \frac{\overset{1}{\cancel{D}}}{x+1}$$

$$\frac{2x^3-2x^2+2x+5}{x^3(x+1)} = \frac{Ax^2(x+1)+Bx(x+1)+C(x+1)+Dx^3}{x^3(x+1)}$$

$$2x^3-2x^2+2x+5 = Ax^2(x+1)+Bx(x+1)+C(x+1)+Dx^3$$

$$2x^3-2x^2+2x+5 = \underline{Ax^3}+\underline{Ax^2}+\underline{Bx^2}+\underline{Bx}+\underline{Cx}+C+\underline{Dx^3}$$

$$2x^3-2x^2+2x+5 = x^3(A+D)+x^2(A+B)+x(B+C)+C$$

match up the coefficients to powers of x

x^3: $2 = A+D \longrightarrow D = 2-A = 2-1 = 1 \longrightarrow \boxed{D=1}$

x^2: $-2 = A+B \longrightarrow A = -2-B = -2+3 = 1 \longrightarrow \boxed{A=1}$

x: $2 = B+C \longrightarrow B = 2-C = -3 \longrightarrow \boxed{B=-3}$

1: $\boxed{5 = C}$

$$\int \frac{x^5-x^4-2x^2+2x+5}{x^4+x^3}\,dx = \int\left[x-2+\frac{1}{x}-\frac{3}{x^2}+\frac{5}{x^3}+\frac{1}{x+1}\right]dx$$

$$= \frac{x^2}{2}-2x+\ln|x|+\frac{3}{x}+\frac{5x^{-3+1}}{-3+1}+\ln|x+1|+C$$

$$= \boxed{\frac{x^2}{2}-2x+\ln|x|+\frac{3}{x}-\frac{5}{2x^2}+\ln|x+1|+C}$$

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \text{convergent}, & \text{if } p>1 \\ \text{divergent}, & \text{if } p\le 1 \end{cases}$$

3. Evaluate the improper integral or show that it is divergent.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+25} = \int_{-\infty}^{0} \frac{dx}{x^2+25} + \int_{0}^{\infty} \frac{dx}{x^2+25}$

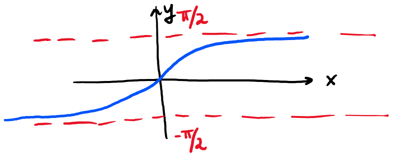
$$= \lim_{t\to-\infty}\int_t^0 \frac{dx}{x^2+25} + \lim_{s\to\infty}\int_0^s \frac{dx}{x^2+25}$$

$$= \lim_{t\to-\infty}\left[\frac15 \arctan\frac x5\right]_t^0 + \lim_{s\to\infty}\left[\frac15 \arctan\frac x5\right]_0^s$$

$$= \frac15\left[\lim_{t\to-\infty}\left(\overset{0}{\cancel{\arctan 0}} - \overset{-\pi/2}{\cancel{\arctan\frac t5}}\right) + \lim_{s\to\infty}\left(\overset{\pi/2}{\cancel{\arctan\frac s5}} - \overset{0}{\cancel{\arctan 0}}\right)\right]$$

$$= \frac15\left(\frac\pi2 + \frac\pi2\right) = \boxed{\frac\pi5}$$

convergent.



(b) $\displaystyle\int_0^\infty \frac{dx}{(x+2)(x+3)} = \lim_{t\to\infty}\int_0^t \frac{dx}{(x+2)(x+3)}$

Partial fractions.

$$\frac{1}{(x+2)(x+3)} = \frac{\overset{1}{\cancel{A}}}{x+2} + \frac{\overset{-1}{\cancel{B}}}{x+3}$$

$$\frac{1}{(x+2)(x+3)} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)} \longrightarrow 1 = A(x+3)+B(x+2)$$

$x=-3$: $1=-B \longrightarrow \boxed{B=-1}$

$x=-2$: $\boxed{1=A}$

$$= \lim_{t\to\infty}\int_0^t\left[\frac{1}{x+2} - \frac{1}{x+3}\right]dx = \lim_{t\to\infty}\Big[\underbrace{\ln|x+2| - \ln|x+3|}_{\text{rewrite as a single ln}}\Big]_0^t$$

$$= \lim_{t\to\infty}\left[\ln\left|\frac{x+2}{x+3}\right|\right]_0^t = \lim_{t\to\infty}\ln\left|\frac{t+2}{t+3}\right| - \ln\frac23$$

L'Hospital's Rule

$$= \ln\left(\lim_{t\to\infty}\frac{t+2}{t+3}\right) - \ln\frac23$$

$$= \ln\left(\lim_{t\to\infty}\frac11\right) - \ln\frac23 = \overset{0}{\cancel{\ln 1}} - \ln\frac23 = -\ln\frac23 = \boxed{\ln\frac32}$$

(c) $\int_1^\infty \frac{\ln x}{x^{3}}dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^3}\,dx$

By parts
$\int u v' dx = uv - \int u' v dx$

$u = \ln x$ $\quad v' = \frac{1}{x^3} = x^{-3}$

$u' = \frac{1}{x}$ $\quad v = \frac{x^{-3+1}}{-3+1} = -\frac{1}{2x^2}$

$$= \lim_{t\to\infty}\left[-\frac{\ln x}{2x^2}\Big|_1^t - \int_1^t \frac{1}{x}\left(-\frac{1}{2x^2}\right)dx \right]$$

$$= \lim_{t\to\infty}\left[-\frac{\ln t}{2t^2} + \frac{\cancelto{0}{\ln 1}}{2} + \frac{1}{2}\int_1^t \frac{dx}{x^3} \right]$$

$$= \lim_{t\to\infty}\left[-\frac{\ln t}{2t^2} + \frac{1}{2}\,\frac{x^{-3+1}}{-3+1}\Big|_1^t \right] = -\lim_{t\to\infty} \overset{\infty/\infty}{\frac{\ln t}{2t^2}} - \frac{1}{4}\lim_{t\to\infty}\left(\cancelto{0}{\frac{1}{t^2}} - 1 \right)$$

L'Hospital's Rule $-\lim_{t\to\infty} \frac{\frac{1}{t}}{4t} + \frac{1}{4}$

$$= -\cancelto{0}{\lim_{t\to\infty} \frac{1}{4t^2}} + \frac{1}{4} = \boxed{\frac{1}{4}}$$

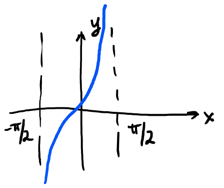
$\tan\frac{\pi}{2} = \infty$

(d) $\int_{\pi/4}^{\pi/2} \tan^2 x\,dx$

$$= \lim_{t\to\frac{\pi}{2}^-} \int_{\pi/4}^t \tan^2 x\,dx = \lim_{t\to\frac{\pi}{2}^-} \int_{\pi/4}^t (\sec^2 x - 1)\,dx$$

$$= \lim_{t\to\frac{\pi}{2}^-} \left(\tan x - x\right)_{\pi/4}^t = \cancelto{\infty}{\lim_{t\to\frac{\pi}{2}^-}(\tan x)} - \tan\frac{\pi}{4} - \lim_{t\to\frac{\pi}{2}^-} t + \frac{\pi}{4}$$

$= \infty$, $\boxed{\text{divergent.}}$



$$\int_0^1 \frac{dx}{x^p} = \begin{cases} \text{convergent, if } p<1 \\ \text{divergent, if } p\geq 1 \end{cases}$$

(e) $\displaystyle\int_1^{10} \frac{dx}{x^2-9} = \int_1^3 \frac{dx}{x^2-9} + \int_3^{10} \frac{dx}{x^2-9}$

$x^2-9=0 \rightarrow x=\pm 3.$

$p=2>1$ divergent.

(f) $\displaystyle\int_1^{\infty} \sin(\pi x)\,dx = \lim_{t\to\infty}\int_1^t \sin(\pi x)\,dx = -\lim_{t\to\infty}\left(\frac{1}{\pi}\cos\pi x\right)\Big|_1^t$

$$= -\frac{1}{\pi}\lim_{t\to\infty}\cos\pi t + \frac{1}{\pi}\cancelto{-1}{\cos\pi}$$

DNE

values of $\cos\pi t$ oscilate between -1 and 1

$\displaystyle\lim_{t\to\infty}\cos\pi t$ DNE.

$\boxed{\text{diverges by oscilation}}$

(g) $\displaystyle\int_0^{5} \frac{dx}{\sqrt[3]{x-5}} = \lim_{t\to5^-}\int_0^t \frac{dx}{(x-5)^{1/3}} = \lim_{x\to5^-} \frac{(x-5)^{-1/3+1}}{-1/3+1}\Big|_0^t$

$p=\frac{1}{3}<1$

$$= \lim_{x\to5^-}\frac{(x-5)^{2/3}}{2/3}\Big|_0^t = \frac{3}{2}\cancelto{0}{\lim_{x\to5^-}(x-5)^{2/3}} - \frac{3}{2}(-5)^{2/3}$$

$$= -\frac{3}{2}(-5)^{2/3} \doteq \boxed{-\frac{3}{2}(25)^{1/3}}$$

$0 < g(x) \leq f(x)$

(a) $\int_1^\infty f(x)dx$ is convergent, then $\int^\infty g(x)dx$ is convergent, too

(b) $\int^\infty g(x)dx$ is divergent, then $\int f(x)$ is divergent.

4. Use the Comparison Test to determine whether the following integral is divergent or convergent.

(a) $\displaystyle\int_1^\infty \frac{dx}{\sqrt[3]{x^3+1}}$

$p=1$

$x^3+1 \leq x^3+x^3$

$x^3+1 \leq 2x^3 \longrightarrow \sqrt[3]{x^3+1} \leq \sqrt[3]{2x^3}$

$\sqrt[3]{x^3+1} \leq x\sqrt[3]{2}$

$$\frac{1}{\sqrt[3]{(x^3+1)}} \geqslant \frac{1}{x \cdot 2^{1/3}}$$

$\int_1^\infty \frac{dx}{x \cdot 2^{1/3}}$ is divergent $(p=1)$

By the Comparison Theorem, $\int_1^\infty \frac{dx}{\sqrt[3]{(x^3+1)}}$ is divergent.

(b) $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}dx$

$p=2>1$

$$\frac{0}{x^2} \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

$\int_1^\infty \frac{dx}{x^2}$ is convergent

$p>1$

By the Comparison Theorem, $\int^\infty \frac{\cos^2 x}{x^2}dx$ will converge.

(c) $\int_1^\infty \frac{2+\cos x}{\boxed{\sqrt{x^4+x^2}}}dx$ ← $p > \frac{4}{2} = 2 > 1$

$$2 - 1 \le 2\cos x \le 1 + 2$$

$$\frac{1}{\sqrt{x^4+x^2}} \le \underbrace{\frac{2+\cos x}{\sqrt{x^4+x^2}}} \le \underbrace{\frac{3}{\sqrt{x^4+x^2}}}$$ ↙ convergent

since $\int_1^\infty \frac{3}{\sqrt{x^4+x^2}} dx$ is convergent $(p = 2 > 1)$

By the Comparison Test, so is $\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}} dx$

$\int_1^\infty \frac{2+\cos x}{\sqrt{x}} dx$ $(p = \frac{1}{2} < 1)$ — divergent

divergent → $\boxed{\frac{1}{\sqrt{x}}} \le \frac{2+\cos x}{\sqrt{x}} \le \frac{3}{\sqrt{x}}$

since $\int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent $(p = \frac{1}{2} < 1) \Longrightarrow$ then $\int_1^\infty \frac{2+\cos x}{\sqrt{x}} dx$ is divergent

(d) $\int_1^\infty \frac{1+e^{-x}}{x} dx$

$$\lim_{x\to\infty} e^{-x} = 0$$

$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$

$\int_1^\infty \frac{dx}{x}$ is divergent, so is $\int_1^\infty \frac{1+e^{-x}}{x} dx$