Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, maclaurin series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

1. Find $f^{(152)}(0)$, the 152nd derivative for the function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n(n+2)}$ $= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+2)} x^n$

$\frac{f^{(n)}(0)}{n!}$

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{3^n(n+2)}$$

$$f^{(n)}(0) = \frac{(-1)^n \cdot n!}{3^n(n+2)}, \quad f^{(152)}(0) = \frac{(-1)^{152} \cdot (152)!}{3^{152}(152+2)} = \boxed{\frac{(152)!}{154 \cdot 3^{152}}}$$

$n=152$

2. Find the Taylor series expansion for the following functions $\sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n$

(a) xe^{3x} centered at $x=5$

$n=0$ $f(x) = xe^{3x}$

$n=1$ $f'(x) = e^{3x} + 3xe^{3x} = e^{3x}(1+3x)$

$n=2$ $f''(x) = 3e^{3x}(1+3x) + 3e^{3x} = 3e^{3x}(2+3x)$

$= 6e^{3x} + 9xe^{3x}$

$n=3$ $f'''(x) = 18e^{3x} + 9e^{3x} + 27xe^{3x}$

$= 9e^{3x}(3+3x)$

$n=4$ $f^{(iv)}(x) = 27\cdot(3)e^{3x} + 27e^{3x} + 27\cdot 3xe^{3x}$

$= 27e^{3x}(4+3x)$

$$f^{(n)}(x) = 3^{n-1}e^{3x}(n+3x)$$

$$f^{(n)}(5) = 3^{n-1}(n+15)e^{15}$$

$$\boxed{xe^{3x} = \sum_{n=0}^{\infty} \frac{3^{n-1}(n+15)e^{15}}{n!}(x-5)^n}$$

(b) $\ln(1+x)$ centered at $x=2$ $\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$

$f(x) = \ln(1+x)$, $f(2) = \ln(1+2) = \ln 3$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -1\cdot(1+x)^{-2}$

$f'''(x) = (-1)(-2)(1+x)^{-3}$

$= (-1)^2 \cdot 2!\,(1+x)^{-3}$

$f^{(iv)}(x) = (-1)(-2)(-3)(1+x)^{-4}$ . . . .

$= (-1)^3 \cdot 3!\,(x+1)^{-4}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$

$f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{3^n}$

$$\ln(1+x) = \ln(1+2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{3^n\, n!}(x-2)^n$$

$$\boxed{\ln(1+x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 3^n}(x-2)^n}$$

$n=0$

$f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{}$

$= (-1)^3 \cdot 3! \, (x+1)^{-4}$

$$f^{(n)}(x) = (-1)^{n-1} (n-1)! \, (1+x)^{-n}$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

Table 1: Important Maclaurin Series and their Radii of Convergence

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \dbinom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!} x^2 + \dfrac{k(k-1)(k-2)}{3!} x^3 + \cdots$	$R = 1$

3. Find the Maclaurin series for $f(x)$.

(a) $f(x) = x^2 \cos(2x)$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} \cdot x^{2n}}{(2n)!}$$

$$x^2 \cos(2x) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} \cdot x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} \cdot x^{2n} \cdot x^2}{(2n)!}$$

$$\boxed{x^2 \cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} \cdot x^{2n+2}}{(2n)!}}$$

(b) $f(x) = xe^{-x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}$$

$$xe^{-x^2} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\boxed{xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}}$$

(c) $f(x) = \sqrt[3]{8+x}$ $= \sqrt[3]{8\left(1+\frac{x}{8}\right)} = \sqrt[3]{8} \cdot \left(1+\frac{x}{8}\right)^{1/3} = 2\left(1+\frac{x}{8}\right)^{1/3}$

$$= 2\left[1 + \frac{1}{3}\left(\frac{x}{8}\right) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{x}{8}\right)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{x}{8}\right)^3 + \dots\right]$$

$$= 2\left[1 + \frac{x}{24} + \frac{\left(-\cancel{2}/9\right)}{\cancel{2}!}\left(\frac{x}{8}\right)^2 + \frac{10/27}{3!}\left(\frac{x}{8}\right)^3 + \dots\right]$$

$$\boxed{= 2\left[1 + \frac{x}{24} - \frac{1}{9}\left(\frac{x}{8}\right)^2 + \frac{5}{81}\left(\frac{x}{8}\right)^3 + \dots\right] = \sqrt[3]{8+x}}$$

$\frac{1}{3}\left(\frac{1}{3}-1\right) = -\frac{2}{9}$

$\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right) = -\frac{2}{9} \cdot \left(-\frac{5}{3}\right)$

$= +\frac{10}{27}$

$\frac{10/27}{3!} = \frac{10}{27} \cdot \frac{1}{2 \cdot 3}$

$= \frac{5}{81}$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

4. Evaluate the integral

(a) $\int 5x^2 \arctan(7x^3)\, dx$

$$\arctan(7x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(7x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1}\cdot (x^3)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1}\, x^{6n+3}}{2n+1}$$

$$x^2 \arctan(7x^3) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1}\, x^{6n+3}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1}\, x^{6n+3}\cdot x^2}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1}\, x^{6n+5}}{2n+1}$$

$$5\int x^2 \arctan(7x^3)\, dx = 5\int \left[\sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1}\, x^{6n+5}}{2n+1}\right] dx$$

$$= 5 \sum_{n=0}^{\infty} \left[\int \frac{(-1)^n\, 7^{2n+1}\, x^{6n+5}}{2n+1}\, dx\right]$$

$$= 5 \sum_{n=0}^{\infty} \left[\frac{(-1)^n\, 7^{2n+1}}{2n+1} \frac{x^{6n+6}}{6n+6}\right] + C = \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n (7^{2n+1})(5)}{2n+1} \frac{x^{6n+6}}{6(n+1)}}$$

(b) $\int_0^x e^{-t^2}\, dt$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$\int_0^x e^{-t^2}\, dt = \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right] dt = \sum_{n=0}^{\infty} \left[\int_0^x \frac{(-1)^n t^{2n}}{n!}\, dt\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\int_0^x t^{2n}\, dt\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{t^{2n+1}}{2n+1}\Big|_0^x$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}}$$

5. Find the sum of the following series

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 3^n \pi^n}{n!} = \sum_{n=2}^{\infty} \frac{(-3\pi)^n}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - 1 - x$$

$x = -3\pi$

$$\sum_{n=2}^{\infty} \frac{(-3\pi)^n}{n!} = e^{-3\pi} - 1 - (-3\pi) = \boxed{e^{-3\pi} - 1 + 3\pi}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} \overset{x=3}{=\!=} \boxed{\sin 3}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(c) $\displaystyle f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=2}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n} \frac{1}{(2n)!}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

$$\underbrace{1}_{n=0} \underbrace{- \frac{x^2}{2}}_{n=1} + \sum_{n=2}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \frac{x^2}{2} - 1 + \cos x \longleftarrow \text{plug in } x = \frac{\pi}{6}$$

$$\sum_{n=2}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n} \frac{1}{(2n)!} = \frac{(\pi/6)^2}{2} - 1 + \cos\frac{\pi}{6} = \boxed{\frac{\pi^2}{128} - 1 + \frac{\sqrt{3}}{2}}$$

$$f(4) + \frac{f'(4)}{1}(x-4) + \frac{f''(4)}{2}(x-4)^2 + \frac{f'''(4)}{6}(x-4)^3 = T_3(x)$$

6. Find the $\boxed{\text{third}}$ degree Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 4$.

$f(x) = x^{1/2}$	$f(4) = 2$
$f'(x) = \frac{1}{2}x^{-1/2}$	$f'(4) = \frac{1}{4}$
$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$	$f''(4) = -\frac{1}{4} \cdot \frac{1}{8} = -\frac{1}{32}$
$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$	$f'''(4) = \frac{3}{8} \cdot \frac{1}{32} = \frac{3}{256}$

$$T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-2)^2 + \frac{3}{256} \cdot \frac{1}{6}(x-4)^3$$

$$\boxed{T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-2)^2 + \frac{1}{512}(x-4)^3}$$

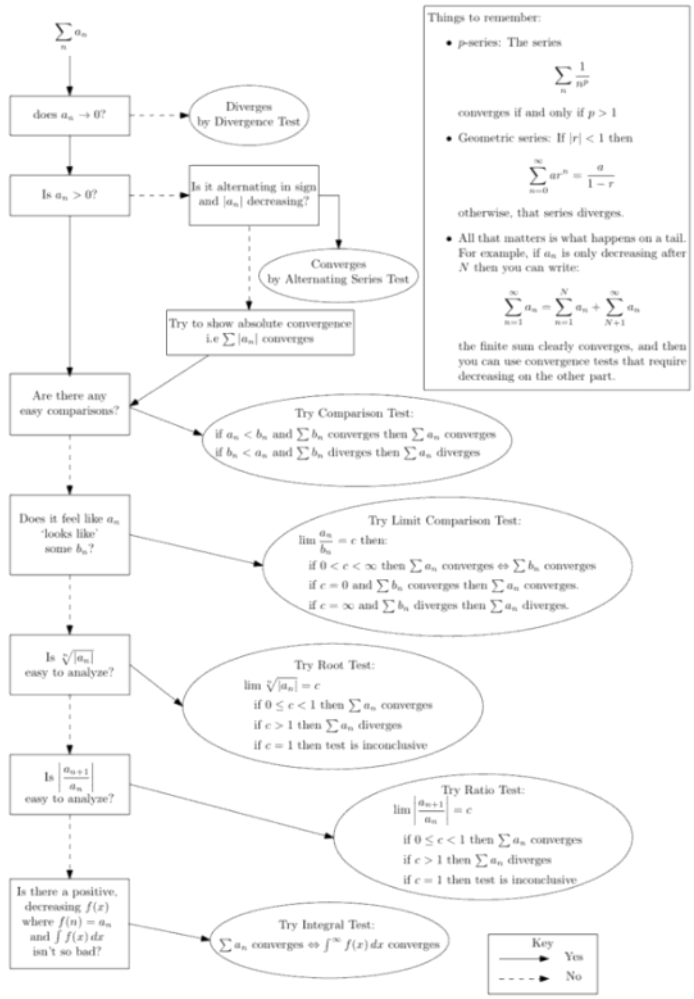
7. Find the $\boxed{\text{second}}$ degree Taylor polynomial for $f(x) = \arctan(x)$ at $x = 1$.

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$f(x) = \arctan x$	$f(1) = \arctan 1 = \frac{\pi}{4}$
$f'(x) = \frac{1}{1+x^2}$	$f'(1) = \frac{1}{1+1} = \frac{1}{2}$
$f''(x) = -\frac{2x}{(1+x^2)^2}$	$f''(1) = -\frac{2}{(1+1)^2} = -\frac{2}{4} = -\frac{1}{2}$

$$T_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{2} \cdot \frac{1}{2}(x-1)^2 = \boxed{\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2}$$

Series Convergence Flowchart



8. Find the radius and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{3^n(x-5)^n}{n^2+1}$

$a_n = \frac{3^n(x-5)^n}{n^2+1}, \quad a_{n+1} = \frac{3^{n+1}(x-5)^{n+1}}{(n+1)^2+1}$

Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-5)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{3^n(x-5)^n}\right|$$

$$= \lim_{n\to\infty} 3|x-5|\cdot\frac{n^2+1}{(n+1)^2+1} = 3|x-5|\lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1} = \boxed{3|x-5| < 1}$$

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$$-1 < 3(x-5) < 1$$

$$-\frac{1}{3} < x-5 < \frac{1}{3}$$

$$5-\frac{1}{3} < x < 5+\frac{1}{3}$$

$$\boxed{\frac{14}{3} < x < \frac{16}{3}} \text{ interval of convergence.}$$

Test the end points. $\boxed{x = \frac{14}{3}}$

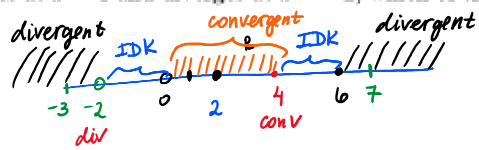
$$\sum_{n=1}^{\infty}\frac{3^n\left(\frac{14}{3}-5\right)^n}{n^2+1} = \sum_{n=1}^{\infty}\frac{3^n\left(-\frac{1}{3}\right)^n}{n^2+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$ – convergent by Alternating series Test.

$\boxed{x = \frac{16}{3}}$ $$\sum_{n=1}^{\infty}\frac{3^n\left(\frac{16}{3}-5\right)^n}{n^2+1} = \sum_{n=1}^{\infty}\frac{3^n\left(\frac{1}{3}\right)^n}{n^2+1} = \sum_{n=1}^{\infty}\frac{1}{n^2+1}$$ convergent by the Integral Test.

$\frac{14}{3} \le x \le \frac{16}{3}$ – interval of convergence.

$R = 1/5$

9. If the power series $\sum_{n=0}^{\infty} c_n(x-2)^n$ converges at $x=4$ and diverges at $x=-2$, which of the following series will also converge?



$R = 2$

$|x-2| < 2$

$0 < x < 4$

(a) $\sum_{n=0}^{\infty} c_n(-1)^n 3^n$ – divergent

(b) $\sum_{n=0}^{\infty} c_n 7^n$ – divergent

(c) $\sum_{n=0}^{\infty} c_n(+1)^n 2^{-n} = \sum_{n=0}^{\infty} \frac{c_n}{2^n} = \sum_{n=0}^{\infty} c_n \left(\frac{1}{2}\right)^n$ — conv.

$$\frac{1}{1-☺} = \sum_{n=0}^{\infty} (☺)^n, \quad |☺| < 1$$

10. Find the power series representation for the function $f(x) = \frac{x^3}{(5-3x^2)^2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\left(\frac{1}{5-3x^2}\right)' = \frac{-6x}{(5-3x^2)^2}$$

$$f(x) = \frac{x^3}{(5-3x^2)^2} = \frac{x^2}{-6} \cdot \frac{-6x}{(5-3x^2)^2} = -\frac{x^2}{6}\left(\frac{1}{5-3x^2}\right)'$$

$$\frac{1}{5-3x^2} = \frac{1}{5} \cdot \frac{1}{1-\frac{3}{5}x^2} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}x^2\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n x^{2n}$$

$$\left(\frac{1}{5-3x^2}\right)' = \left(\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n x^{2n}\right)' = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n (x^{2n})'$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n (2n) x^{2n-1}$$

$$-\frac{x^2}{6} \cdot \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n (2n) x^{2n-1} = -\frac{1}{30}\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n (2n)\, x^{2n-1} \cdot x^2$$

$$= \boxed{-\frac{1}{15}\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n \cdot n \cdot x^{2n+1}}$$

11. Which of the following series converge. State the test you have used.

(a) $\sum_{n=2}^{\infty} \frac{n^2-2n-1}{n^2+4n}$ Divergent by the Divergence Test.

$$\lim \frac{n^2-2n-1}{n^2+4n} = 1 \neq 0$$

(b) $\sum_{n=0}^{\infty} \frac{1}{n^2+2n+4}$

Convergent by the limit comparison Test when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (p-series, $p=2>1$)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+2n+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+4} = 1$$

(c) $\sum_{n=1}^{\infty} ne^{-n^2}$ convergent by the Integral Test.

converges $-\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx$ $\left| \begin{array}{l} -x^2=u \\ du=-2x\,dx \\ u(1)=-(1)^2=-1 \\ u(t)=-t^2 \end{array} \right| \rightarrow x\,dx=-\frac{du}{2}$

$$= \lim_{t\to\infty}\int_{-1}^{-t^2} e^u \frac{du}{-2} = -\frac{1}{2}\lim_{t\to\infty} e^u \Big|_{-1}^{-t^2} = -\frac{1}{2}\left[\lim_{t\to\infty} \cancel{e^{-t^2}} - e^{-1}\right] = \frac{1}{2}.$$

0

(d) $\sum_{n=0}^{\infty} \frac{1}{n!}$

Ratio Test, $a_n = \frac{1}{n!}$, $a_{n+1} = \frac{1}{(n+1)!}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{(n+1)!}\cdot\frac{n!}{1} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

$\sum_{n=0}^{\infty}\frac{1}{n!}$ converges by the Ratio Test.

(e) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ - convergent, p- series $p = 3/2 > 1$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges, p- series $p = 1/3 < 1$

If $\Sigma |a_n|$ is convergent $\rightarrow \Sigma a_n$ is convergent absolutely.

12. Which of the following series converges absolutely?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ - convergent by the Alternating Series Test

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ - divergent $p = 1/2 < 1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ $\leftarrow$ converges by the alternating series test. | $\frac{\ln(n+1)}{n+1} < \frac{\ln n}{n}$ $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n}$ — diverges.

Integral Test $\int_2^{\infty} \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_2^{t} \frac{\ln x}{x} dx$ | $u = \ln x$, $du = \frac{dx}{x}$, $u(2) = \ln 2$, $u(t) = \ln t$ |

$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u\, du = \lim_{t\to\infty} \frac{u^2}{2} \Big|_{\ln 2}^{\ln t} = \frac{1}{2} \Big[\lim_{t\to\infty} \underbrace{(\ln^2 t)}_{\infty} - \ln^2 2 \Big] = \infty$

$\boxed{\text{conditionally convergent}}$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^4}$ — absolutely convergent.

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n^4} \right| = \sum_{n=2}^{\infty} \frac{1}{n^4}$ — con vergent, $p = 4 > 1$

13. How many terms is needed to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ within 2×10^{-9}?

$\sum_{n=1}^{\infty} \frac{1}{n^5}$ is convergent by the Integral Test.

$$S_n < \int_n^{\infty} f(x)dx, \quad f(x) = \frac{1}{x^5}$$

$$\int_n^{\infty} \frac{1}{x^5} dx = \left[\frac{x^{-5+1}}{-5+1}\right]_n^{\infty} = -\frac{1}{4x^4}\Big|_n^{\infty} = \frac{1}{4n^4} \leq 2\times 10^{-9}$$

$$\frac{1}{4n^4} = \frac{2}{10^9}$$

$$4n^4 > \frac{10^9}{2} \quad \text{or} \quad n^4 > \frac{10^9}{8}$$

$$n > \sqrt[4]{\frac{10^9}{8}} \approx 125.74$$

$$\boxed{n = 126}$$

alternating series.

14. Use the fifth partial sum to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+3)!}$. Find the upper bound for the error in the estimate.

$$S \approx S_5 = -\frac{1^2}{(1+3)!} + \frac{2^2}{(2+3)!} - \frac{3^2}{(3+3)!} + \frac{4^2}{(4+3)!} - \frac{5^2}{(5+3)!}$$

$$|R_5| < |b_6|$$

$$R_5 < \frac{6^2}{(6+3)!} = \frac{\cancel{36}}{\cancel{9}\cdot 8\cdot 7\cdot 6\cdot 5\cdot \cancel{4}\cdot 3\cdot 2\cdot 1} = \frac{1}{8\cdot 7\cdot 6\cdot 5\cdot 3\cdot 2}$$