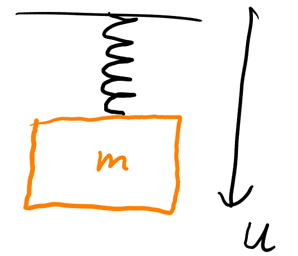
3.7–3.8 MECHANICAL VIBRATIONS

Review

- Standard equation for a **mass and spring system**

$$mu'' + \gamma u' + ku = F(t)$$

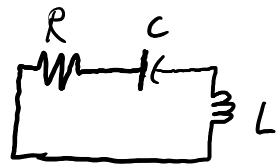


$u(t)$ = position of mass from equilibrium

m = mass, γ = damping coefficient, k = spring constant, $F(t)$ = external force

- Standard equation for **electronic circuits**

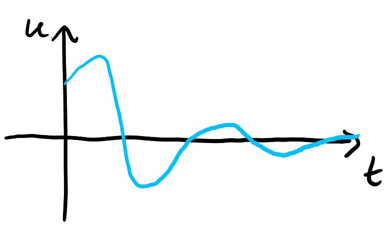
$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$



$Q(t)$ = charge on the capacitor

L = inductance, R = resistance, C = capacitance, $E(t)$ = impressed voltage

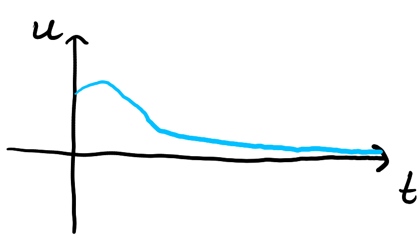
- Damping
  - Underdamped



complex roots

$$u(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$$

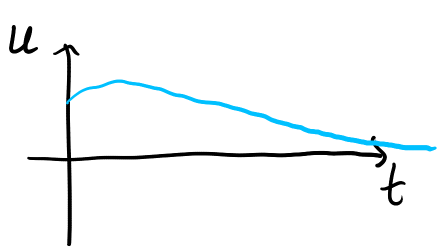
  - Critically damped



repeated roots

$$u(t) = c_1 e^{rt} + c_2 t e^{rt}$$

  - Overdamped



distinct roots

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

- To find the amplitude of a solution, use the following formula.

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta),$$

where $R = \sqrt{A^2 + B^2}$ and $\delta = \arctan(B/A)$. The **amplitude** is R, and the **angular frequency** is ω.

- The **period** is $2\pi/$(angular frequency).
- **Natural frequency:** The frequency of the system when there is no damping and no external force.
- **Resonance** occurs when there is no damping and the external force of a system matches the natural frequency of the system. When this happens, the oscillations grow unboundedly large.

Exercise 1

A mass of 3 kg is attached to a spring. When the mass is attached, the spring stretches an additional 50 cm. There is no damping. The mass is initially stretched an additional 10 cm and pushed with an initial downward velocity of 2 m/s. Find the position of the mass over time. Find the frequency, period, and amplitude. (Use $g = 10\,\mathrm{m/s^2}$.)

Formula to remember: $kL = mg$.

$$k = \frac{mg}{L} = \frac{(3\,\mathrm{kg})(10\,\mathrm{m/s^2})}{0.5\,\mathrm{m}} = 60\,\mathrm{N/m}$$

$$3u'' + 60u = 0, \quad u(0) = 0.1\,\mathrm{m}, \quad u'(0) = 2\,\mathrm{m/s}$$

$$3r^2 + 60 = 0$$

$$r^2 = -20$$

$$r = \pm i\sqrt{20}$$

$$u(t) = c_1\cos(\sqrt{20}\,t) + c_2\sin(\sqrt{20}\,t)$$

$$u'(t) = -\sqrt{20}\,c_1\sin(\sqrt{20}\,t) + \sqrt{20}\,c_2\cos(\sqrt{20}\,t)$$

Solve for c_1 and c_2.

$$u(0) = c_1 = 0.1$$

$$u'(0) = \sqrt{20}\, c_2 = 2 \Rightarrow c_2 = \frac{2}{\sqrt{20}}$$

$$u(t) = \frac{1}{10}\cos(\sqrt{20}\,t) + \frac{2}{\sqrt{20}}\sin(\sqrt{20}\,t)$$

Convert to other form to find amplitude:

$$u(t) = \underbrace{\sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{2}{\sqrt{20}}\right)^2}}_{\text{amplitude}} \cos\left(\underbrace{\sqrt{20}}_{\text{angular freq}}\,t + \tan^{-1}\left(\frac{2}{10\sqrt{20}}\right)\right)$$

$$\text{amplitude} = \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{2}{\sqrt{20}}\right)^2} \text{ meters}$$

$$\text{frequency} = \sqrt{20} \text{ rad/s}$$

$$\text{period} = \frac{2\pi}{\sqrt{20}} \text{ seconds}$$

Exercise 2

A spring has a mass attached to it with weight 5 N. When the spring is attached, the spring stretches 200 cm. When the mass is moving 2 m/s, there is a damping force of 4 N. We start the spring from the equilibrium position with an initial upward velocity of 3 m/s. Find the location of the mass over time. (Use $g = 10\,\mathrm{m/s^2}$.)

Find m, γ, k.

$$m = \frac{\text{weight}}{g} = \frac{5N}{10\,m/s^2} = \frac{1}{2}\,kg$$

$$\gamma = \frac{|\text{Force}|}{\text{speed}} = \frac{4N}{2\,m/s} = 2\,\frac{N}{m/s}$$

$$k = \frac{mg}{L} = \frac{5N}{2\,m} = 5/2\ N/m$$

Initial value problem:

$$\frac{1}{2}u'' + 2u' + 5/2\,u = 0, \qquad u(0) = 0\,m, \quad u'(0) = -3\,m/s$$

Solve for u.

$$\frac{1}{2}r^2 + 2r + 5/2 = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(5)}}{2} = -2 \pm \frac{\sqrt{16-20}}{2} = -2 \pm i\frac{\sqrt{4}}{2} = -2 \pm i$$

$$u(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$$

Solve for c_1 and c_2.

$$u(0) = c_1 = 0$$

$$u(t) = c_2 e^{-2t} \sin(t)$$

$$u'(t) = -2c_2 e^{-2t} \sin(t) + c_2 e^{-2t} \cos(t)$$

$$u'(0) = C_2 = -3$$

$$\boxed{u(t) = -3e^{-2t} \sin(t) \text{ meters}}$$

Exercise 3

A spring has a mass of 1 kg attached to it with a spring constant of 4 N/m. There is an external force $\sin(2t)$ pushing on the mass. Suppose the mass starts from equilibrium at rest. Find the location of the mass over time. What happens as $t \to \infty$?

$m = 1$ kg

$k = 4$ N/m

$\gamma = 0$

$$u'' + 4u = \sin(2t), \quad u(0) = 0, \quad u'(0) = 0.$$

Solve for u.

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

$$u_h(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

Guess: $u_p(t) = At\sin(2t) + Bt\cos(2t)$

$$u_p'(t) = A\sin(2t) + 2At\cos(2t) + B\cos(2t) - 2Bt\sin(2t)$$

$$= (A - 2Bt)\sin(2t) + (2At + B)\cos(2t)$$

$$u_p''(t) = -2B\sin(2t) + 2(A - 2Bt)\cos(2t) + 2A\cos(2t) - 2(2At + B)\sin(2t)$$

$$= (-4At - 4B)\sin(2t) + (4A - 4Bt)\cos(2t)$$

plug into diff eq:

$$\left(\cancel{-4At}-4B\right)\sin(2t)+\left(4A\cancel{-4Bt}\right)\cos(2t)$$

$$\cancel{+4At\sin(2t)}\cancel{+4Bt\cos(2t)}=\sin(2t)$$

$$\underbrace{-4B}_{=1}\sin(2t)+\underbrace{4A}_{=0}\cos(2t)=\sin(2t)$$

$$4B=1 \qquad 4A=0$$

$$B=1/4 \qquad A=0$$

$$u_p(t)=\frac{1}{4}t\cos(2t)$$

General solution:

$$u(t)=c_1\cos(2t)+c_2\sin(2t)+\frac{1}{4}t\cos(2t)$$

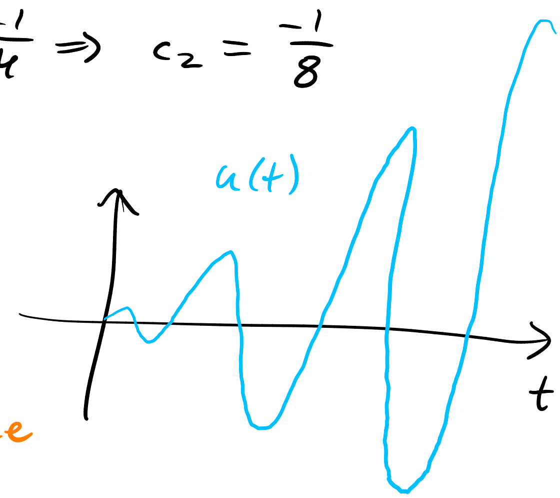
Solve for c_1 and c_2:

$$u'(t)=-2c_1\sin(2t)+2c_2\cos(2t)+\frac{1}{4}\cos(2t)-\frac{1}{2}t\sin(2t)$$

$$u'(0)=2c_2+\frac{1}{4}=0 \Rightarrow 2c_2=\frac{-1}{4} \Rightarrow c_2=\frac{-1}{8}$$

$$u(0)=c_1=0$$

$$\boxed{u(t)=\frac{-1}{8}\sin(2t)+\frac{1}{4}t\cos(2t)}$$



resonance

Exercise 4

Consider the equation

$$3u'' + \gamma u' + 10u = 0.$$

Find the value of γ that makes the system critically damped.

Idea: Critically damped means repeated roots.

$$3r^2 + \gamma r + 10 = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(3)(10)}}{2}$$

want stuff under square root to be 0 to get repeated roots

$$\gamma^2 - (4)(3)(10) = 0$$

$$\gamma^2 = 120$$

$$\boxed{\gamma = \sqrt{120}}$$

want $+\sqrt{120}$ because it's unphysical for γ to be negative.

6.1: DEFINITION OF LAPLACE TRANSFORM

Review

- The Laplace transform is defined by $\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$.
- For many functions, you can just look up the Laplace transform in the table.

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n $(n = 1, 2, ...)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n$ $(n = 1, 2, ...)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

- The Laplace transform is also linear: $\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}$.
- To take the **inverse Laplace transform**, you can also use the table. However, if your function does not match the things in the table, then you need to first do partial fractions.
- Partial fractions review
  - Simple roots

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

  - Irreducible quadratics

$$\frac{1}{(s^2+3)(s-1)} = \frac{As+B}{s^2+3} + \frac{C}{s-1}$$

  - Repeated roots

$$\frac{1}{(s^2+3)^2(s-1)^3} = \frac{As+B}{(s^2+3)^2} + \frac{Cs+D}{s^2+3} + \frac{E}{(s-1)^3} + \frac{F}{(s-1)^2} + \frac{G}{s-1}$$

Exercise 5

Using the definition of the Laplace transform, compute the Laplace transform of $3t$.

$$\mathcal{L}\{3t\} = \int_0^\infty e^{-st} \cdot 3t\,dt$$

$u = 3t \qquad dv = e^{-st}dt$

$du = 3 \qquad v = \frac{-1}{s}e^{-st}$

$$= \frac{-3}{s} t e^{-st}\Big|_0^\infty - \int_0^\infty \frac{-3}{s} e^{-st}\,dt$$

$$= -\frac{3}{s}\Big(\cancel{\lim_{t\to\infty} t e^{-st}} - 0\Big) - \frac{3}{s^2} e^{-st}\Big|_0^\infty$$

0 if $s>0$

$$= -\frac{3}{s^2}\Big(\cancel{\lim_{t\to\infty} e^{-st}} - 1\Big) = \boxed{\frac{3}{s^2} \quad \text{for} \quad s>0}$$

0 if $s>0$

Exercise 6

Find the Laplace transform of $\begin{cases} 3 & 0 \le t \le 5, \\ e^{2t} & t > 5. \end{cases}$

Exercise 7

Find the Laplace transform of $g(t) = 6t^4 - 3\cos(2t) + e^{4t}\sin(t)$.

$$G(s) = 6\mathcal{L}\{t^4\} - 3\mathcal{L}\{\cos(2t)\} + \mathcal{L}\{e^{4t}\sin(t)\}$$

$$\boxed{= 6\frac{4!}{s^5} - 3\frac{s}{s^2+4} + \frac{1}{(s-4)^2+1}}$$

Exercise 8

Find the inverse transform of $F(s) = \dfrac{2}{s^4}$.

$$F(s) = \frac{2}{6}\frac{6}{s^4}$$

$$\boxed{f(t) = \frac{1}{3}t^3}$$

Exercise 9

Find the inverse transform of $G(s) = \dfrac{3}{s^2+3}$.

$$G(s) = \frac{3}{\sqrt{3}} \frac{\sqrt{3}}{s^2+3}$$

$$\boxed{g(t) = \frac{3}{\sqrt{3}} \sin(\sqrt{3}\,t)}$$

Exercise 10

Find the inverse transform of $Y(s) = \dfrac{2}{s^2+4s+7}$.

$$s = \frac{-4 \pm \sqrt{16-4(7)(1)}}{2}$$

Complex roots, so irreducible quadratic

$$Y(s) = \frac{2}{s^2+4s+4+3} = \frac{2}{(s+2)^2+3}$$

$$= \frac{2}{\sqrt{3}} \frac{\sqrt{3}}{(s+2)^2+3}$$

$$\boxed{y(t) = \frac{2}{\sqrt{3}} e^{-2t} \sin(\sqrt{3}\,t)}$$

Exercise 11

Find the inverse transform of $F(s) = \dfrac{s+1}{(s-2)(s^2+2s+3)}$.

$$s = \frac{-2 \pm \sqrt{4-4(3)(1)}}{2}$$

complex roots, so irreducible quadratic

$$= \frac{A}{s-2} + \frac{Bs+C}{s^2+2s+3}$$

$$s+1 = A(s^2+2s+3) + (Bs+C)(s-2)$$

$$= As^2 + 2As + 3A + Bs^2 - 2Bs + Cs - 2C$$

$$= \underbrace{(A+B)}_{=0}s^2 + \underbrace{(2A-2B+C)}_{=1}s + \underbrace{3A-2C}_{=1}$$

$B = -A$

$B = -3/11$

$2A - 2(-A) + C = 1$

$4A + C = 1$

$C = 1 - 4A$

$C = 1 - \frac{4 \cdot 3}{11}$

$= -1/11$

$3A - 2C = 1$

$3A - 2(1-4A) = 1$

$11A - 2 = 1$

$A = 3/11$

$$F(s) = \frac{3}{11}\left(\frac{1}{s-2}\right) + \frac{-\frac{3}{11}s - \frac{1}{11}}{s^2+2s+3}$$

$$= \frac{3}{11}\left(\frac{1}{s-2}\right) - \frac{1}{11}\left(\frac{3s+1}{(s^2+2s+1)+2}\right)$$

$$= \frac{3}{11}\left(\frac{1}{s-2}\right) - \frac{1}{11}\,\frac{3(s+1-1)+1}{(s+1)^2+2}$$

$$= \frac{3}{11}\left(\frac{1}{s-2}\right) - \frac{1}{11}\left(\frac{3(s+1)}{(s+1)^2+2} - \frac{2}{(s+1)^2+2}\right)$$

$$= \frac{3}{11}\left(\frac{1}{s-2}\right) - \frac{3}{11}\left(\frac{s+1}{(s+1)^2+2}\right) + \frac{2}{11\sqrt{2}}\left(\frac{\sqrt{2}}{(s+1)^2+2}\right)$$

$$\boxed{f(t) = \frac{3}{11}e^{2t} - \frac{3}{11}e^{-t}\cos(\sqrt{2}t) + \frac{2}{11\sqrt{2}}e^{-t}\sin(\sqrt{2}t)}$$

Exercise 12

Find the inverse transform of $Y(s) = \frac{s}{(s-1)^2(s+2)}$. $= \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+2}$

$$s = A(s+2) + B(s-1)(s+2) + C(s-1)^2$$

$s=1$: $1 = 3A \Rightarrow A = 1/3$

$s=-2$: $-2 = 9C \Rightarrow C = -2/9$

$s=0$: $0 = 2A - 2B + C = 2/3 - 2B - 2/9 \Rightarrow 2B = \frac{6}{9} - \frac{2}{9} = \frac{4}{9}$

$\Rightarrow B = 2/9$

$$Y(s) = \frac{1}{3}\left(\frac{1}{(s-1)^2}\right) + \frac{2}{9}\left(\frac{1}{s-1}\right) - \frac{2}{9}\left(\frac{1}{s+2}\right)$$

$$\boxed{y(t) = \frac{1}{3}te^t + \frac{2}{9}e^t - \frac{2}{9}e^{-2t}}$$