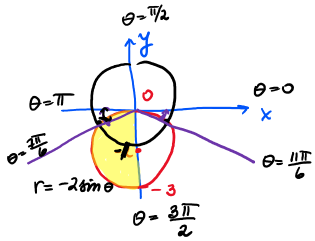
$$A = \frac{1}{2}\int_a^b r^2(\theta)\, d\theta$$

1. Find the area inside the curve $r = -2\sin\theta$ in the third quadrant. At what angles will the above curve intersect with the polar curve $r = 1$?

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

$r = -2\sin\theta$
$r^2 = -2r\sin\theta$
$x^2 + y^2 = -2y$
$x^2 + (y^2 + 2y) = 0$
$x^2 + (y+1)^2 = 1$



$\pi \le \theta \le \frac{3\pi}{2}$

$$A = \frac{1}{2}\int_{\pi}^{3\pi/2} r^2(\theta)\, d\theta = \frac{1}{2}\int_{\pi}^{3\pi/2} (-2\sin\theta)^2\, d\theta = \frac{1}{2}\int_{\pi}^{3\pi/2} (4\sin^2\theta)\, d\theta$$

$$= 2\int_{\pi}^{3\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta = \left(\theta - \frac{1}{2}\sin 2\theta\right)_{\pi}^{3\pi/2}$$

$$= \frac{3\pi}{2} - \pi - \frac{1}{2}\sin 3\pi + \frac{1}{2}\sin 2\pi = \boxed{\frac{\pi}{2}}$$

$r = 1 \leftrightarrow x^2 + y^2 = 1$, $r = -2\sin\theta$

$1 = -2\sin\theta$, $\sin\theta = -\frac{1}{2}$, $\theta_1 = \pi + \frac{\pi}{6} = \boxed{\frac{7\pi}{6}}$

$\theta_2 = 2\pi - \frac{\pi}{6} = \boxed{\frac{11\pi}{6}}$

1

2. Find the area of the region enclosed by one loop of the curve $r = 4\cos 3\theta$.

$r = 4\cos 3\theta$

$\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} + \pi n$, $n = 0, \pm 1, \pm 2, \ldots$

$\theta = \frac{\pi}{6} + \frac{\pi n}{3}$

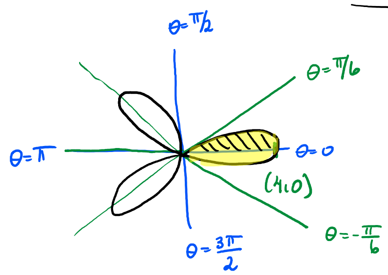
$n=0 \rightarrow \theta = \frac{\pi}{6}$

$n=1 \rightarrow \theta = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

$n=2 = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$

$n=3 = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$

$n=4 = \frac{\pi}{6} + \frac{4\pi}{3} = \frac{9\pi}{3} = 3\pi$



$\cos 3\theta = 1$

$3\theta = 2\pi n$, $n = 0, \pm 1, \ldots$

$\theta = \frac{2\pi n}{3}$

$\theta = 0$, $\theta = \frac{4\pi}{3}$

$\theta = \pi$

$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

$2 \cdot \frac{1}{2}\int_0^{\pi/6} (4\cos 3\theta)^2 d\theta$ $A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (4\cos 3\theta)^2 \, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} 16\cos^2 3\theta \, d\theta = 8\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2} \, d\theta = \ldots$

3. Find the area of the region inside the curve $r = 3\cos\theta$ and outside the curve $r = 1 + \cos\theta$.

a circle — cardioid.

$(r)\ r = 3r\cos\theta$

$r^2 = 3r\cos\theta$

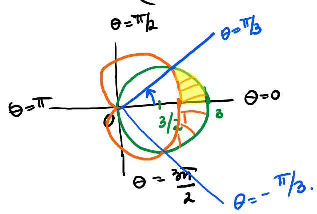
$x^2 + y^2 = 3x$

$x^2 - 3x + y^2 = 0$

$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}.$

$r = 1 + \cos\theta$

symmetry about x-axis.



intersection

$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}$

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} \left[(3\cos\theta)^2 - (1+\cos\theta)^2 \right] d\theta$$

$$= \int_0^{\pi/3} \left(9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta \right) d\theta$$

$$= \int_0^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta \right) d\theta$$

$$= \int_0^{\pi/3} \left(8 \cdot \frac{1+\cos 2\theta}{2} - 1 - 2\cos\theta \right) d\theta = \ldots$$

Remember

$$\int_0^{2\pi} \sin x \, dx = \int_0^{2\pi} \cos x \, dx = 0$$

4. Evaluate the integral

(a) $\int t^2 \cos(1-t^3)dt$ $\left| \begin{array}{l} u = 1-t^3 \\ du = -3t^2 dt \quad \rightarrow \quad t^2 dt = -\frac{du}{3} \end{array} \right| = -\frac{1}{3}\int \cos u \, du = -\frac{1}{3}\sin u + C$

$$= \boxed{C - \frac{1}{3}\sin(1-t^3)}$$

(b) $\int \frac{x^2}{\sqrt{1-x}}dx$ $\left| \begin{array}{l} u = 1-x \quad \longrightarrow \quad x = 1-u \\ dx = -du \end{array} \right| = \int \frac{\overbrace{(1-u)^2}^{x^2}}{\underbrace{\sqrt{u}}_{\sqrt{1-x}}} \underbrace{(-du)}_{dx}$

$$= -\int \frac{1-2u+u^2}{\sqrt{u}} du = -\int \left(u^{-1/2} - 2u^{1/2} + u^{3/2}\right) du = \ldots$$

(c) $\int x^3 e^{x^2} dx = \int x \cdot x^2 e^{x^2} dx$ $\left| \begin{array}{l} u = x^2 \\ du = 2x\,dx \longrightarrow x\,dx = \frac{du}{2} \end{array} \right|$

$= \frac{1}{2}\int u e^u \, du$ By parts.

	D	I
+	u	e^u
−	1	e^u
	0	e^u

$= \frac{1}{2}(ue^u - e^u) + C$

$= \boxed{\frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C}$

(d) $\int (x^3 + 2x^2 - x)e^{3x}\, dx$

	D	I
+	x^3+2x^2-x	e^{3x}
−	$3x^2+4x-1$	$\frac{1}{3}e^{3x}$
+	$6x+4$	$\frac{1}{9}e^{3x}$
−	6	$\frac{1}{27}e^{3x}$
	0	$\frac{1}{81}e^{3x}$

$$= \frac{1}{3}(x^3+2x^2-x)\, e^{3x} - (3x^2+4x-1)\cdot\frac{1}{9}e^{3x}$$

$$+ (6x+4)\cdot\frac{1}{27}e^{3x} - \frac{6}{81}e^{3x} + C$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(e) $\int \frac{\ln x}{x^2} dx$

$u = \ln x \quad v' = \frac{1}{x^2}$

$u' = \frac{1}{x} \quad v = -\frac{1}{x}$

$$= -\frac{1}{x}\ln x - \int \frac{1}{x}\left(-\frac{1}{x}\right)dx$$

$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = \boxed{-\frac{\ln x}{x} - \frac{1}{x} + C}$$

(f) $\underbrace{\int e^{3x}\sin(2x)\,dx}_{I} = \left| \begin{matrix} u=\sin 2x & v'=e^{3x} \\ u'=2\cos 2x & v=\frac{1}{3}e^{3x} \end{matrix} \right| = \frac{1}{3}\sin 2x\, e^{3x} - \frac{2}{3}\int \cos 2x\, e^{3x}\,dx \left| \begin{matrix} u=\cos 2x & v'=e^{3x} \\ u'=-2\sin 2x & v=\frac{1}{3}e^{3x} \end{matrix} \right|$

$$= \frac{1}{3}\sin 2x\, e^{3x} - \frac{2}{3}\left[\frac{1}{3}e^{3x}\cos 2x - \frac{1}{3}(-2)\int \sin 2x\, e^{3x}\,dx \right]$$

$$= \frac{1}{3}\sin 2x\, e^{3x} - \frac{2}{9}e^{3x}\cos 2x - \frac{4}{9}\underbrace{\int \sin 2x\, e^{3x}\,dx}_{I}$$

$$I = \frac{1}{3}\sin 2x\, e^{3x} - \frac{2}{9}e^{3x}\cos 2x - \frac{4}{9}I$$

$$I + \frac{4}{9}I = \frac{1}{3}e^{3x}\sin 2x - \frac{2}{9}e^{3x}\cos 2x$$

$$\int \sin 2x\, e^{3x}\,dx = I = \boxed{\frac{9}{13}\left(\frac{1}{3}e^{3x}\sin 2x - \frac{2}{9}e^{3x}\cos 2x\right) + C}$$

(g) $\int_0^{\pi/8} \sin^2(2x)\cos^3(2x)\,dx \left| \begin{matrix} u=\sin(2x) \\ du=2\cos 2x\,dx \\ \cos^2(2x)=1-\sin^2 2x \\ u(0)=0 \\ u(\frac{\pi}{8})=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}} \end{matrix} \right| = \frac{1}{2}\int_0^{1/\sqrt{2}} u^2(1-u^2)\,du = \ldots$

(h) $\int \sin^2 x \cos^4 x \, dx = \int \frac{1-\cos 2x}{2} \cdot \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \frac{1}{8} \int (1-\cos 2x)(1+2\cos 2x + \cos^2 2x) dx$

$= \frac{1}{8} \int [1 + 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x] dx$

$= \frac{1}{8} \int [1 + \cos 2x - \cos^2 2x - \cos^3 2x] dx$

$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x\right) - \frac{1}{8}\int \cos^2 2x \, dx - \frac{1}{8}\int \underbrace{\cos^3(2x)}_{\cos(2x)\cos^2(2x)} dx$ | $u = \sin 2x$, $du = 2\cos 2x \, dx$

$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x\right) - \frac{1}{8}\int \frac{1+\cos 4x}{2} dx - \frac{1}{16}\int (1-u^2) du = \ldots$

(i) $\int_0^{\pi/4} \tan^4 x \sec^4 x \, dx = \int_0^{\pi/4} \tan^4 x (\sec^2 x)\underbrace{(\sec^2 x) dx}_{du} =$ | $u = \tan x$, $du = \sec^2 x \, dx$, $\sec^2 x = 1 + \tan^2 x = 1 + u^2$, $u(0) = 0$, $u(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

$= \int_0^1 u^4 (u^2+1) du = \ldots$

(j) $\int \tan^3 x \sec^3 x \, dx = \int (\tan x \sec x) \tan^2 x \sec^2 x \, dx$ | $u = \sec x$, $du = \sec x \tan x \, dx$, $\tan^2 x = \sec^2 x - 1 = u^2 - 1$

$= \int (u^2-1) u^2 \, du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$

(k) $\int (4x^2-25)^{-3/2}\,dx = \int \frac{dx}{(4x^2-25)^{3/2}}$

$2x = 5\sec t$

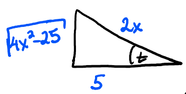
$x = \frac{5\sec t}{2}, \quad dx = \frac{5}{2}\sec t\tan t\,dt$

$4x^2-25 = 25\sec^2 t - 25 = 25\tan^2 t$

$$= \int \frac{\frac{5}{2}\sec t\tan t\,dt}{\left(\sqrt{25\tan^2 t}\right)^3} = \frac{5}{2}\int \frac{\sec t\tan t\,dt}{125\tan^3 t} = \frac{1}{50}\int \frac{\sec t\,dt}{\tan^2 t}$$

$$= \frac{1}{50}\int \frac{\frac{1}{\cos t}}{\frac{\sin^2 t}{\cos^2 t}}\,dt = \frac{1}{50}\int \frac{\cos t}{\sin^2 t}\,dt = \frac{1}{50}\int \frac{du}{u^2} \quad [u = \sin t]$$

$$= -\frac{1}{50}\left(\frac{1}{u}\right) + C = -\frac{1}{50}\cdot\frac{1}{\sin t} + C$$



$\sec t = \frac{2x}{5}$

$\cos t = \frac{1}{\sec t} = \frac{5}{2x}$

$\sin t = \frac{\sqrt{4x^2-25}}{2x}$

$$= -\frac{1}{50}\frac{2x}{\sqrt{4x^2-25}} + C$$

(l) $\int \frac{(x-1)^2}{5\sqrt{24-x^2+2x}}dx$

$24-x^2+2x = 24-(x^2-2x) = 24-(x^2-2x+1)+1$

$= 25-(x-1)^2$

$t = \arcsin\left(\frac{x-1}{5}\right)$

$\int \frac{(x-1)^2}{5\sqrt{25-(x-1)^2}}dx$

$x-1 = 5\sin t \rightarrow \sin t = \frac{x-1}{5}$

$dx = 5\cos t\, dt$

$\sqrt{25-(x-1)^2} = 5\cos t \rightarrow \cos t = \frac{\sqrt{25-(x-1)^2}}{5}$

$= \int \frac{25\sin^2 t\,(5\cos t\, dt)}{5\cdot 5\cos t}$

$= 5\int \sin^2 t\, dt = 5\int \frac{1-\cos 2t}{2}dt = 5\left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)+C$

$= \frac{5}{2}t - \frac{5}{4}(2\sin t\cos t) + C$

$= \frac{5}{2}\arcsin\left(\frac{x-1}{5}\right) - \frac{5}{2}\,\frac{x-1}{5}\cdot\frac{\sqrt{25-(x-1)^2}}{5} + C$

(m) $\int \frac{5x^2+x+12}{x^3+4x}dx$

Partial fraction.

$$\frac{5x^2+x+12}{x^3+4x} = \frac{5x^2+x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{3}{x} + \frac{2x+1}{x^2+4}$$

$$\frac{5x^2+x+12}{x^3+4x} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)}$$

$$5x^2+x+12 = x^2(A+B) + Cx + 4A$$

x^2: $5 = A+B \longrightarrow \boxed{B = 5-A = 2}$

x: $\boxed{1 = C}$

1: $12 = 4A \longrightarrow \boxed{A=3}$

$$\int \frac{5x^2+x+12}{x^3+4x}dx = \int\left[\frac{3}{x} + \frac{2x+1}{x^2+4}\right]dx$$

$$= 3\int\frac{dx}{x} + 2\int\frac{x\,dx}{x^2+4} + \int\frac{dx}{x^2+4}$$

$\left| \begin{array}{l} u = x^2+4 \\ du = 2x\,dx \end{array} \right|$

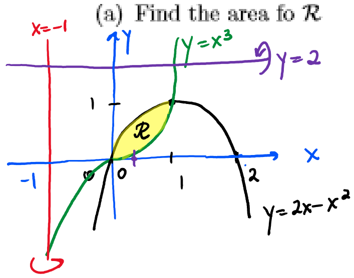
$$= 3\ln|x| + \int\frac{du}{u} + \frac{1}{2}\arctan\frac{x}{2} + C$$

$$= 3\ln|x| + \ln|u| + \frac{1}{2}\arctan\frac{x}{2} + C$$

$$= \boxed{3\ln|x| + \ln|x^2+4| + \frac{1}{2}\arctan\frac{x}{2} + C}$$

5. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$.

(a) Find the area fo $\mathcal{R}$



$x^3 = 2x - x^2$

$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x+2)(x-1) = 0.$

$x_1 = 0, \; x_2 = -2, \; x_3 = 1.$

$y = -(x^2 - 2x) = -(x^2 - 2x + 1) + 1$

$y = 1 - (x-1)^2$

$y = x(2-x) = 0 \longrightarrow x = 0, \; x = 2.$

$A = \int_0^1 (2x - x^2 - x^3)\, dx = \ldots$

(b) Find the volume obtained by rotating $\mathcal{R}$ about the line $x = -1$.

shells.

$V_{x=-1} = 2\pi \int_0^1 \underbrace{(x+1)}_{r} \underbrace{(2x - x^2 - x^3)}_{h} \, dx = \ldots$

(c) Find the volume obtained by rotating $\mathcal{R}$ about the line $y = 2$.

washers.

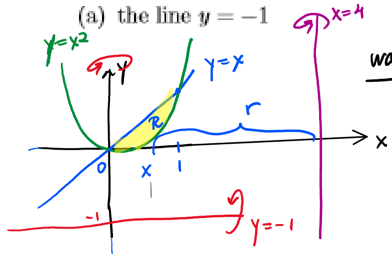
inner radius $= 2 - 2x + x^2$

outer radius $= 2 - x^3$

$V_{y=2} = \pi \int_0^1 \left[(2 - x^3)^2 - (2 - 2x + x^2)^2 \right] dx = \ldots$

6. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about

(a) the line $y = -1$



washers.

inner radius = x^2+1

outer radius = $x+1$

$V_{y=-1} = \pi \int_0^1 \left[(x+1)^2 - (x^2+1)^2 \right] dx = \ldots$

(b) the y-axis

shells.

radius = x

height = $x - x^2$

$V_y = 2\pi \int_0^1 x(x - x^2)\, dx = \ldots$

(c) the line $x = 4$

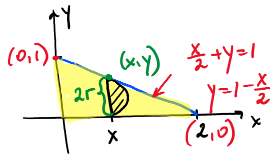
shells

$r = 4 - x$

$h = x - x^2$

$V_{x=4} = 2\pi \int_0^1 (4-x)(x-x^2)\, dx = \ldots$

7. The base of solid S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.



$A = \frac{\pi r^2}{2}$

$0 \le x \le 2.$

integrate for x

$2r = y$

$r = \frac{y}{2}$

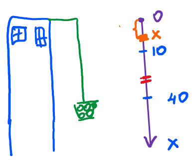
$r = \frac{1}{2}\left(1-\frac{x}{2}\right)$

$A(x) = \frac{\pi}{2} \cdot \frac{1}{4}\left(1-\frac{x}{2}\right)^2$

$= \frac{\pi}{8}\left(1-\frac{x}{2}\right)^2$

$$V = \int_0^2 A(x)\,dx = \frac{\pi}{8}\int_0^2 \left(1-\frac{x}{2}\right)^2 dx = \ldots$$

8. A cable 40 feet long weighing 6 pounds per foot is hanging off the side of a 50 foot tall building. At the bottom of the cable is a bucket of rocks weighing 100 pounds. How much work is required to pull 10 feet of the cable to the top of the building?



$0 \le x \le 40.$

$0 \le x \le 10$

weight = 6 lb/ft.

dist = x

$10 \le x \le 40.$

weight = 6 lb/ft.

dist = 10 ft

$$W = \int_0^{10} 6x\,dx + \int_{10}^{30} 60\,dx + \underbrace{100(10)}_{\text{bucket}} = \ldots$$

9. A spring has a natural length of 20 cm. If a 10 J work is required to keep it stretched to a length 25 cm, how much work is done in stretching the spring from 30 cm to 80 cm?

20 cm → 0

25 cm → 25 − 20 = 5 cm = 0.05 m

$f(x) = kx$

$$10 = \int_0^{0.05} kx\,dx$$

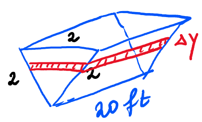
$$10 = k\frac{x^2}{2}\Big|_0^{0.05} \rightarrow 20 = k\frac{1}{400} \rightarrow \boxed{k = 8000}$$

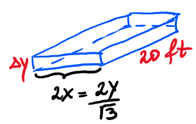
30 → 30 − 20 = 10 cm = 0.1 m

80 → 80 − 20 = 60 cm = 0.6 m

$$W = \int_{0.1}^{0.6} (8000)x\,dx = 4000x^2\Big|_{0.1}^{0.6} = 4000(0.36 - 0.01) = \ldots$$

10. A tank of water is 20 ft long and has a vertical cross section in a shape of an equilateral triangle with sides 2 ft long. The tank is filled with water to a depth of 18 inches. Determine the amount of work needed to pump all of the water to the top of the tank. The weight of water is 62.5 lb/ft^3 .

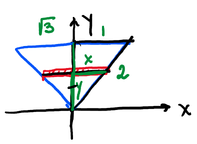




integrate for y

$0 \le y \le \frac{18}{12} = \frac{3}{2}$

$0 \le y \le \frac{3}{2}$



similar triangles

$$\frac{x}{1} = \frac{y}{\sqrt{3}} \Rightarrow x = \frac{y}{\sqrt{3}}$$

volume of the slice $V = (2x)(20)\Delta y = \frac{2y}{\sqrt{3}}(20)\Delta y$

weight of the slice $= V(62.5)$

$$W = \frac{40y}{\sqrt{3}}(62.5)\Delta y$$

distance traveled

$d = \sqrt{3} - y$

$$\text{Work done} = \int_0^{3/2} (\sqrt{3} - y)\frac{40y}{\sqrt{3}}(62.5)\,dy = \ldots$$

11. Find the average value of $f = \sin^2 x \cos x$ on $[-\pi/2, \pi/2]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{ave} = \frac{1}{\frac{\pi}{4}+\frac{\pi}{2}}\int_{-\pi/2}^{\pi/4} \sin^2 x \cos x\,dx$$

$u = \sin x$
$du = \cos x\,dx$
$u(-\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) = -1$
$u(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$= \frac{4}{3\pi}\int_{-1}^{1/\sqrt{2}} u^2\,du = \ldots$$

12. Write out the form of the partial fraction decomposition (do not try to solve)

$$\frac{20x^3+12x^2+x}{(x^3-x)(x^3+2x^2-3x)(x^2+x+1)(x^2+9)^2}$$

irreducible irreducible

$$(x^3-x)(x^3+2x^2-3x) = x(x^2-1)\,x(x^2+2x-3) = x^2(x-1)(x+1)(x+3)(x-1)$$
$$= x^2(x-1)^2(x+1)(x+3)$$

$$= \frac{20x^3+12x^2+x}{x^2(x-1)^2(x+1)(x+3)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{x+1} + \frac{F}{x+3} + \frac{Gx+H}{x^2+x+1} + \frac{Ix+J}{x^2+9} + \frac{Kx+L}{(x^2+9)^2}$$

13. Determine whether the given integral is convergent or divergent.

(a) $\int_1^\infty \frac{4+\cos^4 x}{x} dx$

$0 \le \cos^4 x \le 1$

$\frac{4}{x} \le \frac{4+\cos^4 x}{x} \le \frac{5}{x}$ (p=1, divergent)

$\int_1^\infty \frac{4+\cos^4 x}{x} dx$ diverges by comparison with $\int_1^\infty \frac{4dx}{x}$

(b) $\int_1^\infty \frac{3+\sin x}{x^2} dx$ (p=2, convergent)

$-1 \le \sin x \le 1$

$\frac{2}{x^2} \le \frac{3+\sin x}{x^2} \le \frac{4}{x^2}$

$\int_1^\infty \frac{3+\sin x}{x^2} dx$ is convergent by comparison with $\int_1^\infty \frac{4}{x^2} dx$.

(c) $\int_0^\infty \frac{1}{\sqrt{x}+e^{4x}} dx$ – convergent by comparison with $\int_0^\infty e^{-4x} dx$.

$\frac{1}{\sqrt{x}+e^{4x}} \le \frac{1}{e^{4x}} = e^{-4x}$

$\int_0^\infty e^{-4x} dx$ is convergent

$\int_0^\infty e^{-4x} dx = \lim_{t\to\infty} \int_0^t e^{-4x} dx = -\frac{1}{4}\left[\lim_{t\to\infty} e^{-4t} - \lim_{t\to\infty} e^0\right] = \frac{1}{4}$ (with $\lim_{t\to\infty} e^{-4t} = 0$)

14. Compute the following integrals or show that they diverge.

(a) $\displaystyle\int_e^\infty \frac{dx}{x\ln^5 x} = \lim_{t\to\infty}\int_e^t \frac{dx}{x\ln^5 x} = \left| \begin{array}{l} u=\ln x \\ du=\frac{dx}{x} \\ u(e)=\ln e = 1 \\ u(t)=\ln t \end{array} \right| = \lim_{t\to\infty}\int_1^{\ln t}\frac{du}{u^5}$

$\displaystyle = \lim_{t\to\infty} \frac{u^{-4}}{-4}\Big|_1^{\ln t} = -\frac{1}{4}\left[\lim_{t\to\infty}\frac{1}{[\ln t]^4} - 1\right] = \frac{1}{4}$, convergent

(b) $\displaystyle\int_{-\infty}^0 (1+x)e^x\,dx = \lim_{t\to-\infty}\int_t^0 (1+x)e^x\,dx$

D	I
$1+x$	e^x
1	e^x
0	e^x

$\displaystyle = \lim_{t\to-\infty}\left[(1+x)e^x - e^x\right]_t^0$

$\infty\cdot 0$

$\displaystyle = \lim_{t\to-\infty}\left(1-(1+t)e^t - e^t\right) = 1 - \lim_{t\to-\infty}(1+t)e^t = 1 - \lim_{t\to-\infty}\frac{1+t}{e^{-t}}$

$\displaystyle = 1 - \lim_{t\to-\infty}\frac{1}{-e^{-t}} = 1 + \lim_{t\to-\infty} e^t = 1$, convergent.

(c) $\displaystyle\int_{-\infty}^\infty \frac{6x^5}{(x^6+3)^3}dx = \int_{-\infty}^0 + \int_0^\infty$

(d) $\displaystyle\int_0^{2020} \frac{1}{\sqrt{2020-x}}dx$