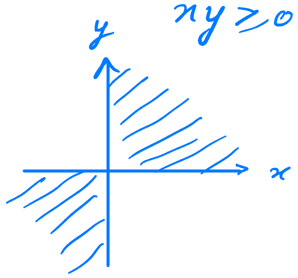
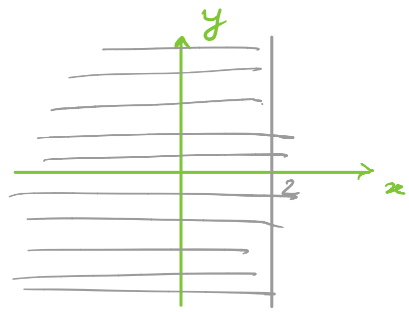
Example 1 (14.1)**.** *Find and sketch the domain of the function.*

(a) $f(x,y) = \sqrt{xy} + e^{\sqrt{2-x}} + 3.$

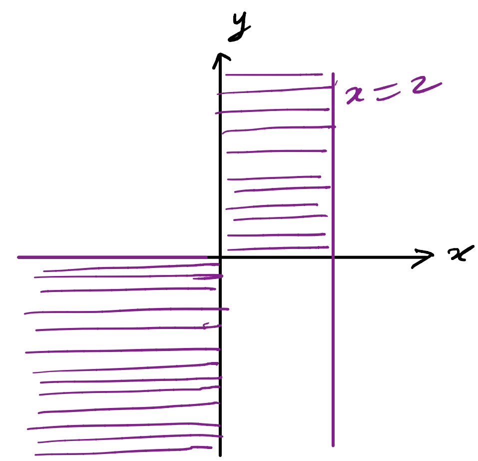
$xy \geq 0$

$x \leq 2$





Domain $= \{(x,y) : xy \geq 0 \text{ and } x \leq 2\}$



(b) $f(x,y) = \dfrac{\sqrt{4 - x^2 - y^2}}{x+y}.$

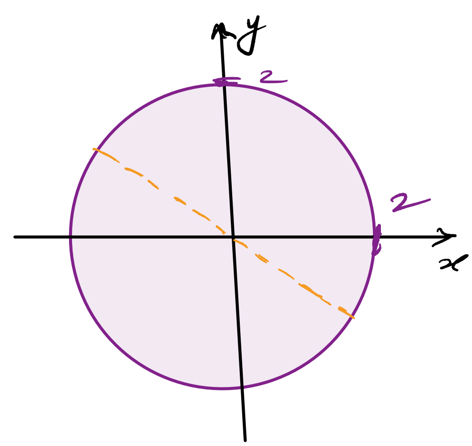
Numerator : $4 - x^2 - y^2 \geq 0$

$\Rightarrow \quad x^2 + y^2 \leq 4$

Denominator : $x + y \neq 0$

$x \neq -y$

Domain $= \{(x,y) : x^2 + y^2 \leq 4 \text{ and } x \neq -y\}$



Example 2 (14.1)**.** *Draw the level curves $z = k$ when $k = 1, 2, 3$ of the function*

$$z = f(x, y) = \ln(x^2 + y^2 - 9),$$

and sketch its graph.

Domain $= \{(x,y): x^2+y^2 > 3^2\}$

$z=1 \Rightarrow \ln(x^2+y^2-9) = 1$

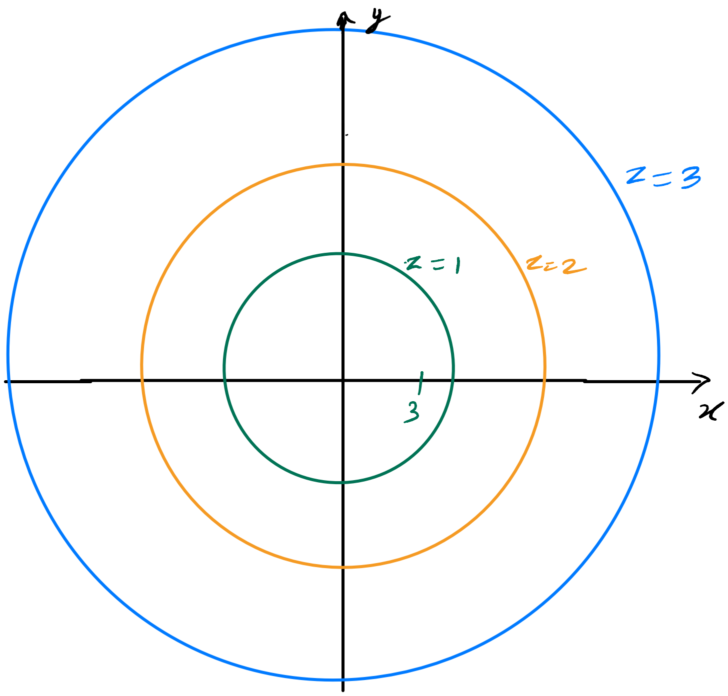
$x^2+y^2 = 9+e$, a circle centered at the origin with radius $\sqrt{9+e}$. This means the height of the function at any point on the circle $x^2+y^2 = 9+e$ is 1.

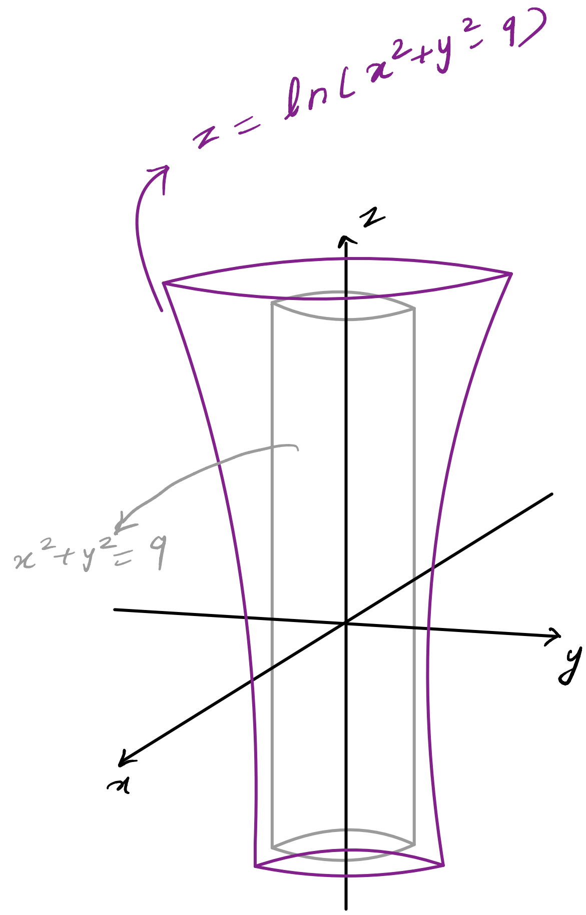
$z=2 \Rightarrow \ln(x^2+y^2-9) = 2$

i.e. $x^2+y^2 = 9+e^2$

$z=3 \Rightarrow \ln(x^2+y^2-9) = 3$

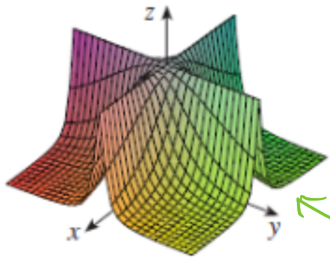
i.e. $x^2+y^2 = 9+e^3$



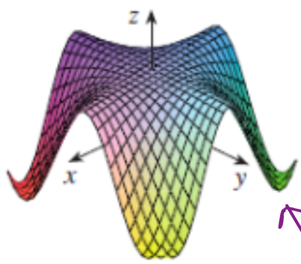


Example 3 (14.1). *Match the graph with its function.*

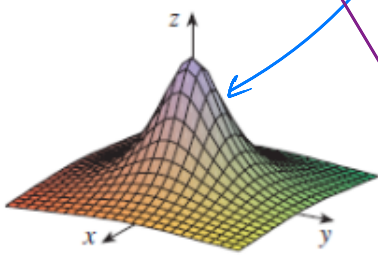
I. (B)



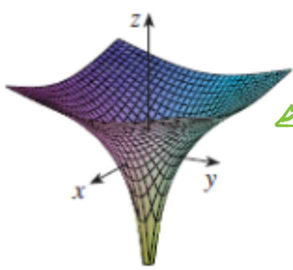
II. (E)



III. (A)



IV. (C)



(A) $f(x,y) = \dfrac{1}{1+x^2+y^2}$ $\quad 0 < z \leq 1$

$z = \frac{1}{2} \Rightarrow x^2+y^2 = 1$, level curves are circles.

(B) $f(x,y) = \dfrac{1}{1+x^2y^2}$ $\quad 0 < z \leq 1.$

$z = 1 \Rightarrow (xy)^2 = 0 \Rightarrow x=0$ or $y=0$.
The height of the function is 1 along x and y-axes.

(C) $f(x,y) = \ln(x^2+y^2)$

$z = k$, a constant implies $x^2+y^2 = e^k$; the level curves are circles. High value of $k \Rightarrow$ bigger circle.

(D) $f(x,y) = \cos\sqrt{x^2+y^2}$ $\quad -1 \leq z \leq 1.$

$z = 0 \Rightarrow \sqrt{x^2+y^2} = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$,
$\Rightarrow$ Level curves are circles.

(E) $f(x,y) = \cos(xy)$ $\quad -1 \leq z \leq 1$

$z = 1 \Rightarrow \cos(xy) = 1$
$xy = 0$ (in particular).
$\Rightarrow x = 0$ or $y = 0$.
The height of the function is 1 along the x- and y-axis.

Example 4 (14.3)**.** *Find the first partial derivative of the function.*

(a) $f(x, y) = \dfrac{\ln x}{(x + y)^2}$

$$f_x(x,y) = \frac{\frac{1}{x}(x+y)^2 - 2(\ln x)(x+y)}{(x+y)^4}$$

$$f_y(x,y) = -2(\ln x)(x+y)^{-3}$$

FTC 1: $\dfrac{d}{dx}\left(\int_a^x f(t)\,dt = f(x)\right)$

a is a constant.

(b) $W(p, q) = \displaystyle\int_p^q \sqrt{1 + t^5}\,dt$

$$\frac{d}{dp} W(p,q) = -\frac{d}{dp}\left(\int_q^p \sqrt{1+t^5}\,dt\right) \overset{FTC\,1}{=} -\sqrt{1+p^5}$$

$$\frac{d}{dq} W(p,q) = \frac{d}{dq}\left(\int_p^q \sqrt{1+t^5}\,dt\right) \overset{FTC\,1}{=} \sqrt{1+q^5}$$

(a) $F(r, s, t) = r\tan(s + 3t) + 2t.$

$$F_r = \tan(s+3t)$$

$$F_s = r\sec^2(s+3t)$$

$$F_t = r(\sec^2(s+3t))(3) + 2$$

$$= 3r\sec^2(s+3t) + 2$$

Example 5 (14.3)**.** *Suppose* $f(x,y)=\sqrt{9-x^2-9y^2}$*. Find* $f_x(2,0)$ *and* $f_y(2,0)$ *and interpret these values as slopes.*

$$f_x=\frac{-\cancel{2}x}{\cancel{2}\sqrt{9-x^2-9y^2}}=\frac{-x}{\sqrt{9-x^2-9y^2}}$$

$$z=\sqrt{9-x^2-9y^2}$$

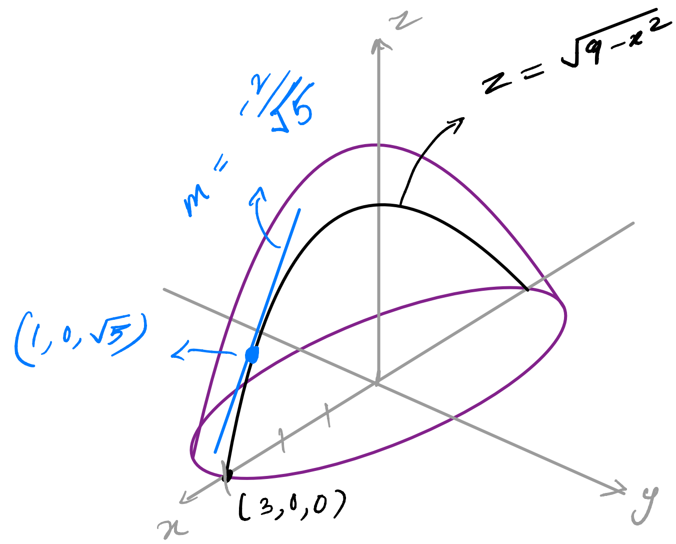
$$x^2+9y^2+z^2=9$$

$$\frac{x^2}{9}+\frac{y^2}{1}+\frac{z^2}{9}=1,$$

The graph of f is the upper-half of the ellipsoid $\frac{x^2}{9}+\frac{y^2}{1}+\frac{z^2}{9}=1$

$$f_x(2,0)=\frac{-2}{\sqrt{5}};$$

This means the slope the of the tangent line at $(2,0,\sqrt{5})$ on the curve $z=\sqrt{9-x^2}$ (which is the intersection of $z=\sqrt{9-x^2-9y^2}$ and $y=0$) is $f_x(2,0)=-\frac{2}{\sqrt{5}}$.



Similarly,

$$f_y=\frac{-9y}{\sqrt{9-x^2-9y^2}}$$

$f_y(2,0)=0$; this means the slope of the tangent line at $(2,0,\sqrt{5})$ on the curve $z=\sqrt{5-9y^2}$ is $f_y(2,0)=0$.

$9y^2+z^2=5$, $z\geq 0$, semi-ellipse.

Example 6 (14.3/14.4). *Suppose* $f(x,y) = e^{xy} - 3x^2y$.

(a) *Find* $f_x(0,-3)$ *and* $f_y(0,-3)$.

(b) *Is* f *differentiable at* $(0,-3)$? *Explain.*

(c) *Find all the second order partial derivatives. Is* f_{xy} *equal to* f_{yx}? *Explain why.*

$f_x = ye^{xy} - 6xy \quad , \quad f_y = xe^{xy} - 3x^2$

(a) $f_x(0,-3) = -3$

$f_y(0,-3) = 0$

(b) Since $f_x(x,y)$ and $f_y(x,y)$ are continuous at $(0,-3)$, f is differentiable at $(0,-3)$.

In fact, f is differentiable everywhere.

(c) $f_{xx} = y^2e^{xy} - 6y \qquad f_{yy} = x^2e^{xy}$

$f_{xy} = (f_x)_y = e^{xy} + xye^{xy} - 6x$

$f_{yx} = (f_y)_x = e^{xy} + xye^{xy} - 6x$

Yes, $f_{xy} = f_{yx}$ as the mixed partial derivatives are continuous.

(Clairauts Theorem).

Tangent Planes *Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

or

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linearization of f at (a, b), and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the **tangent plane approximation** *of f at (a, b).*

Example 7 (14.4). *Find an equation of the tangent plane at the point $P(4, 1, 2)$ on the surface $z = \ln(x - 3y) + 2y$.*

$x_0 \; y_0 \; z_0$

$f_x = \dfrac{1}{x-3y}$ $\quad f_x(4,1) = 1$

$f_y = \dfrac{-3}{x-3y} + 2 \Rightarrow f_y(4,1) = -1$

An equation of the plane is

$z - 2 = 1(x-4) + (-1)(y-1)$

$z - 2 = x - y - 3$

$\boxed{-x + y + z = -1}$

Example 8 (14.4)**.** *Consider the function*

$$f(x,y,z)=\sqrt{x^2+y^2+z^2}.$$

(a) Find the differential of the function.

(b) Find the linearization of the function at the point $(2,1,2)$.

(c) Use the differential or linearization to estimate $\sqrt{1.9^2+1.1^2+2.01^2}$.

(a) $dw = f_x\,dx + f_y\,dy + f_z\,dz$

$= \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,dx + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,dy + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,dz$

(b) $f_x(2,1,2) = \frac{2}{3}$, $f_y(2,1,2) = \frac{1}{3}$, $f_z(2,1,2) = \frac{2}{3}$

$f(2,1,2) = 3$

$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$

$= 3 + \frac{2}{3}(x-2) + \frac{1}{3}(y-1) + \frac{2}{3}(z-2)$

$= \frac{1}{3}[2x + y + 2z]$

(c) $\sqrt{1.9^2+1.1^2+2.01^2} = f(1.9, 1.1, 2.01) \simeq L(1.9, 1.1, 2.01)$

$= \frac{1}{3}[3.8 + 1.1 + 4.02]$

$= \frac{8.92}{3} = 2.973$

Example 9 (14.4)**.** *If $z = x^2 + 2y^2 - x$ and (x, y) changes from $(2, 3)$ to $(1.9, 3.01)$, compare the values Δz and dz.*

$dz = f_x\,dx + f_y\,dy = (2x-1)\,dx + 4y\,dy$.

Given that $(x, y) = (2, 3)$, $(x+\Delta x, y+\Delta y) = (1.9, 3.01)$

So, $dx = \Delta x = -0.1$, $dy = \Delta y = 0.01$

So, $dz = (4-1)(-0.1) + 4(3)(0.01)$

$= -0.30 + 0.12 = -0.18$

$\Delta z = f(1.9, 3.01) - f(2, 3)$

$= [(1.9)^2 + 2(3.01)^2 - 1.9] - [2^2 + 2(3^2) - 2]$

$= 19.83 - 20$

$= -0.17$

$dz \simeq \Delta z$.

The Chain Rule *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 10 (14.5). *The length l, width w, and height h of a rectangular box are changing with respect to time t. At a certain instance the dimensions are $l = 5$ cm, $w = 4$ cm and $h = 10$ cm, and length and width are increasing at a rate of 4 cm/s while the height is decreasing at a rate of 3 cm/s. At that instant find the rate at which the volume of the box is changing.*

$V = lwh$, $l = l(t)$, $w = w(t)$, $h = h(t)$

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\cdot\frac{dl}{dt} + \frac{\partial V}{\partial w}\cdot\frac{dw}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$
$$= wh\frac{dl}{dt} + lh\frac{dw}{dt} + lw\frac{dh}{dt}$$

Given that $l = 5$, $w = 4$, $h = 10$, $dl = dw = 4$, $dh = -3$

so, $$\frac{dV}{dt} = 40(4) + 50(4) + 20(-3)$$
$$= 160 + 200 - 60$$
$$= 300 \text{ cm}^3/s.$$

Example 11 (14.5). *If $z = \tan^{-1}(x^2 + y^2)$, $x = s\ln t$, $y = te^s$, find $\dfrac{\partial z}{\partial s}$ when $s = 2$ and $t = 1$.*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= \frac{2x}{1+(x^2+y^2)^2}\cdot \ln t + \frac{2y}{1+(x^2+y^2)^2}\cdot te^s$$

$s = 2$, $t = 1 \Rightarrow x = 0$, $y = e^2$

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(2,1)} = 0 + \frac{2e^2}{1+(0+e^4)^2}\cdot e^2 = \frac{2e^4}{1+e^8}$$

Example 12 (14.5). *Find* $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$ *if* $x^3 + y^3 + z^3 = 3 - xyz$.

Define $F(x,y,z) = x^3 + y^3 + z^3 + xyz - 3$. Then the level surface $F(x,y,z) = 0$ produces the given surface.

$$F_x = 3x^2 + yz$$

$$F_y = 3y^2 + xz$$

$$F_z = 3z^2 + xy$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(3x^2 + yz)}{3z^2 + xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(3y^2 + xz)}{3z^2 + xy}$$