Math 308: Week-in-Review 7 (3.7-3.8, 6.1-6.2)

3.7-3.8 Mechanical Vibrations

Review

- Standard equation: m is the mass, c is the damping coefficient and k the spring constant
$$mu'' + cu' + ku = F(t)$$
- Amplitude formula: if $u(t) = A\cos(\omega t) + B\sin(\omega t)$ the amplitude R is given by
$$R = \sqrt{A^2 + B^2}$$
- Critical damping:
$$c = 2\sqrt{mk}$$
- Resonance occurs when forcing frequency matches natural frequency

6.1 Laplace Transform

Review

- Definition:
$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)dt$$
- Key transforms:
$$\begin{aligned} e^{at} &\Rightarrow \frac{1}{s-a} \\ t^n &\Rightarrow \frac{n!}{s^{n+1}} \\ e^{at}\cos(bt) &\Rightarrow \frac{s-a}{(s-a)^2+b^2} \end{aligned}$$
- Linearity:
$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

1. A 2.5 kg mass stretches a spring by 45 cm. With damping coefficient $c = 1.8$ N·s/m, the mass is displaced by 12 cm upwards and released with an initial velocity of 1.2 m/s downwards. Find the position function, frequency, period, and amplitude. ($g = 9.8$ m/s^2 and assume that downward is positive.)

2. A 7 N weight stretches a spring by 180 cm. When moving at 2.4 m/s, the damping force is 3.6 N. Starting from 20 cm below the equilibrium position with an upward velocity of 1.8 m/s, find the position function. ($g = 9.8$ m/s^2)

3. A 1 kg mass-spring system with spring constant $k = 9$ N/m experiences forcing $\sin(3t)$ pushing on the mass. Starting at rest from equilibrium, find the steady-state amplitude and the phase shift, assuming no damping. What happens as $t \to \infty$?

4. For $3u'' + cu' + 15u = 0$, determine the value of the damping constant c required for critical damping.

6.1: DEFINITION OF LAPLACE TRANSFORM

Review

- The Laplace transform is defined by

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

- For many functions, you can just look up the Laplace transform in the table.

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n (n = 1, 2, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

- The Laplace transform is also linear:

$$\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}$$

- To take the inverse Laplace transform, you can also use the table. However, if your function does not match the things in the table, then you need to first do partial fractions.
- Partial fractions review
  - Simple roots
  - Irreducible quadratics
  - Repeated roots

5. Compute $\mathcal{L}\{5t^2\}$ using integration by parts.

6. Find $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 2 & 0 \le t < 4 \\ e^{-t} & t \ge 4 \end{cases}$.

7. Compute $\mathcal{L}\{3t^5 - 4\cos(\pi t) + e^{-2t}\sin(3t)\}$.

8. Find $\mathcal{L}^{-1}\left\{\frac{4}{s^6}\right\}$.

9. Determine $\mathcal{L}^{-1}\left\{\dfrac{5s}{s^2+5}\right\}$.

10. Compute $\mathcal{L}^{-1}\left\{\dfrac{3}{(s+1)^2+9}\right\}$.

11. Solve $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{(s+2)(s^2+4s+8)}\right\}$.

12. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s-3)^3(s+1)}\right\}$.