CHARACTERIZING DIFFERENTIAL EQUATIONS

Review

- The **order** of a differential equation is the order of the highest derivative.
- Ordinary vs partial differential equations
  - A **ordinary** differential equation has derivatives with respect to one variable.
  - A **partial** differential equation has derivatives with respect to more than one variable.
- Linear ODEs
  - A **linear** ODE has the form

    $$a_n(x)y^{(n)}(x) + ... + a_1(x)y'(x) + a_0(x)y(x) = g(x).$$

    Said another way, it satisfies the following conditions:
    * All the y's are in different terms.
    * None of the y's are inside a function or to a power.
    * The y's can be multiplied by a function of x.
    * There can be terms that depend only on x.
- Homogeneous linear ODEs
  - A linear ODE is **homogeneous** if the $g(t)$ term is 0.
- Separable ODEs
  - An ODE is **separable** if you can write it in the form $y' = f(x)g(y)$.
- Autonomous ODEs
  - An ODE is **autonomous** if the dependent variable (x) does not show up explicitly. i.e., if x does not show up outside of y.

Exercise 1

Classify the following differential equations. In particular, put it into one (or more) of the following categories and state the order.

- Partial differential equation
- Ordinary differential equation
  - Separable
  - Linear
    - Homogeneous
  - Autonomous

1. $y^2 - y'' + 6 = 0$

2. $f_x - f_y = xf$

3. $y'(x) + x^2 y(x) = 3y(x)$

4. $g' = x^2 \sin(g)$

5. $\sin(x)w''' + w - 3 = 0$

6. $u''(x) = \sin(u(x))$

7. $f^{(5)} - \cos(x^2)f''' - \tan(x)f = 3\tan(x)$

SOLVING DIFFERENTIAL EQUATIONS

Review

- First order ODEs
  - You do **NOT** need to guess which method to use to solve a 1st order ODE!
  - How to determine which method to use:
    1. Is the equation **separable**?
       If yes, use separation of variables.
    2. Is the equation **linear**?
       If yes, use the method of integrating factors.

    2'. Is it a Bernoulli equation[1]?
       If yes, then use $v = y^{1-n}$.

    3. Is the equation **exact**?
       If yes, then use the method for exact equations.

    3'. Is it a homogenous equation[2]?
       If yes, then use $v = y/x$ to get a separable equation.

    4. If none of the above, then try to find an integrating factor to make the equation exact.[3]
- Second order linear ODEs
  - Homogeneous with constant coefficients
    1. Look for solutions of the form $y(t) = e^{rt}$.
    2. Find the characteristic equation.
    3. Find the roots of the characteristic equation.
    4. The general solution is given by
       * Distinct real roots: $c_1e^{r_1t} + c_2e^{r_2t}$
       * Complex roots: $c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$
       * Repeated real roots: $c_1e^{rt} + c_2te^{rt}$
    5. If you have initial conditions, use them to solve for c_1 and c_2.
  - Nonhomogeneous
    * Method of undetermined coefficients (if constant coefficients and you can guess)
    * Variation of parameters

[1] A Bernoulli equation has the form $y' + p(t)y = q(t)y^n$. Not all instructors cover this. You can find examples of Bernoulli equations in Section 2.4 of the textbook, #23–25.

[2] This is NOT the same as the homogeneous linear equations that are covered in Chapter 3. The terminology is confusing. "Homogeneous equation" here refers to a 1st order ODE that can be written in the form $y' = f(\frac{y}{x})$. Not all instructors cover this. You can find examples of these in Section 2.2 of the textbook, #25–31.

[3] Not all instructors cover making an equation exact by using an integrating factor.

Exercise 2

Find the general solution to

$$t^2y' + ty - t = 0.$$

Exercise 3

Solve the initial value problem

$$u' - tu^{-2} = 0, \quad u(1) = -1.$$

Exercise 4

Find the general solution to

$$f'' = 3f' - 2f.$$

Exercise 5

Find the general solution to

$$w'' + 4w' + 4w = 5e^t.$$

Exercise 6

Find the general solution to

$$(4x - 2y)y' + 4y = -2x.$$

Exercise 7

Find the general solution to

$$3g'' - 2g' + 4 = 0.$$

Exercise 8

Solve the initial value problem

$$f = -\frac{1}{9}f'', \quad f(0) = -2, \quad f'(0) = 1.$$

Exercise 9

Find a that makes the equation exact.

$$x^3 + y^a + 2xyy' = 0.$$

Exercise 10

Solve by first finding an integrating factor that makes the equation exact.

$$y + (2xy - e^{-2y})y' = 0.$$

Exercise 11

Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 5y' + 6y = 4e^{-2t} + 3t^3.$$

What is an appropriate guess for the particular solution y_p?

Exercise 12

Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 2y' + y = 3e^t - t\sin(t).$$

What is an appropriate guess for the particular solution y_p?

Exercise 13

Given that x^2 and x^{-1} are solutions to the corresponding homogeneous equation, find a particular solution to

$$x^2y'' - 2y = 3x^2 - 1, \quad x > 0.$$

ANALYSIS OF ODES

Review

- Where is a solution valid?
  - Solution is valid on a **single interval** where the solution is a function that is defined and differentiable.
- Existence and uniqueness
  - **1st order linear ODEs:** If p and g are continuous on an interval $I = (a, b)$ containing the initial condition t_0, then the initial value problem

    $$y' + p(t)y = g(t), \qquad y(t_0) = y_0$$

    has a unique solution on I.
  - **1st order nonlinear ODEs:** Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $(a, b) \times (c, d)$ containing the point (t_0, y_0). Then, there is a unique solution to the initial value problem

    $$y' = f(t, y), \qquad y(t_0) = y_0$$

    on a sufficiently small interval $I_h = (t_0 - h, t_0 + h)$ around t_0.
  - **2nd order linear ODEs:** Consider the initial value problem

    $$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

    If p, q, and g are continuous on an open interval $I = (a, b)$ that contains the point t_0, then there is exactly one solution to the initial value problem and the solution exists throughout the entire interval I.
- The **Wronskian** of y_1 and y_2 is defined by

  $$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

- $\{y_1, y_2\}$ is a **fundamental set of solutions** means that the general solution is $c_1y_1 + c_2y_2$.
- Slope fields
- Equilibrium solutions
- Stability of equilibrium solutions
  - **(Asymptotically) stable:** If you start near it, you go in towards it.
  - **Unstable:** If you start near it, you go away from it.
  - **Semistable:** If you start near on one side, you go towards it, but if you start near on the other side, you go away from it.
- Phase line diagrams

Exercise 14

Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$y' - t^2 \tan(t) y = \sqrt{4-t}, \quad y(0) = \pi.$$

Exercise 15

Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$(t-1)w'' + w' - \ln(t+3)w = t^3 \cos(t), \quad w(2) = -2 \quad w'(2) = 7.$$

Exercise 16

For which values t_0 and y_0 is the following initial value problem guaranteed to have a unique solution?

$$t^2y^2 - (t+y)y' = 0, \quad y(t_0) = y_0.$$

Exercise 17

Show that x and xe^x form a fundamental set of solutions to

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0.$$

Exercise 18

Solve for the explicit solution $u(x)$. Where is the solution to the initial value problem valid? How does this depend on a?

$$u' = u^2, \quad u(0) = a.$$

Exercise 19

Consider the differential equation

$$f' = f(f-2)^2(f-4)$$

(a) Find the equilibrium solutions

(b) Draw the phase line diagram

(c) Sketch the slope field

(d) Determine the stability of each equilibrium solution

(e) Determine $\lim_{t\to\infty} f(t)$ for different initial values $f(0)$.