# 2024 Fall Math 140 Week-In-Review

## Week 11: Sections 5.7 and 5.8

Some Key Words and Terms: Function Transformations, Function Arithmetic, Function Composition, One-to-One Functions, Exponential and Logarithmic Form, Properties of Logarithms, Logarithmic Function, Solving Exponential and Logarithmic Equations, Exponential Models.

Function Transformations:

| Parent Function | Horizontal Shift | Vertical Shift | Reflection across $x$-axis | Vertical Expansion | Vertical Compression |
|---|---|---|---|---|---|
| $f(x)$ | $f(x+a)$ | $f(x)+a$ | $-f(x)$ | $af(x)$ $a>1$ | $\frac{1}{a}f(x)$ $a>1$ |
| $x^2$ | $(x\pm a)^2$ $-a$, right $+a$, left | $x^2\pm a$ $-a$, down $+a$, up | $-x^2 \to -(x^2)$ | $3x^2$ vertical expansion by 3 | $\frac{1}{3}x^2$ vertical compression by 3 |
| $x^3$ | $(x\pm a)^3$ | $x^3\pm a$ | $-x^3 \to -(x^3)$ | $3x^3$ | $\frac{1}{3}x^3$ |
| $\sqrt{x}$ | $\sqrt{x\pm a}$ | $\sqrt{x}\pm a$ | $-\sqrt{x}$ | $3\sqrt{x}$ | $\frac{1}{3}\sqrt{x}$ |
| $\sqrt[3]{x}$ | $\sqrt[3]{x\pm a}$ | $\sqrt[3]{x}\pm a$ | $-\sqrt[3]{x}$ | $3\sqrt[3]{x}$ | $\frac{1}{3}\sqrt[3]{x}$ |
| $\lvert x\rvert$ | $\lvert x\pm a\rvert$ | $\lvert x\rvert\pm a$ | $-\lvert x\rvert$ | $3\lvert x\rvert$ | $\frac{1}{3}\lvert x\rvert$ |
| $b^x$ | $b^{(x\pm a)}$ | $b^x\pm a$ | $-b^x \to -(b^x)$ | $3\cdot b^x$ cannot generally combine these | $\frac{1}{3}\cdot b^x$ |
| $\left(\frac{1}{b}\right)^x$ | $\left(\frac{1}{b}\right)^{(x\pm a)}$ | $\left(\frac{1}{b}\right)^x\pm a$ | $-\left(\frac{1}{b}\right)^x$ | $3\cdot\left(\frac{1}{b}\right)^x$ | $\frac{1}{3}\cdot\left(\frac{1}{b}\right)^x$ |

exponential functions, the "inside" is the power

**Function Arithmetic:** add/subtract/multiply/divide functions

☆ don't over think this & don't mix it up w/ function composition

$(f+g)(x) \rightarrow$ add $f(x)$ & $g(x)$ & simplify

$(f-g)(x) \rightarrow$ subtract $f(x)$ & $g(x)$ (order does matter)

$(fg)(x) \rightarrow$ multiply $f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) \rightarrow$ divide $f(x)$ & $g(x)$ aka $\frac{f(x)}{g(x)}$ ← ☆ we could now have a new restriction

**Function Composition:** plug functions in to other functions

$(f \circ g)(x) \rightarrow f(g(x)) \rightarrow$ plug-in $g(x)$ for all the $x$ in $f(x)$ & simplify

$(g \circ f)(x) \rightarrow g(f(x)) \rightarrow$ plug-in $f(x)$ for all the $x$ in $g(x)$

Generally 3 ways:
1. given $f(x)$ & $g(x)$, compute the composition
2. given a table of value, compute specific value
3. given graphs, compute a specific value

End of 5.7

De-compose: give you the inside of $f(g(x))$ & the result, then ask for $f(x)$

Begin of 5.8

**One-to-One Functions:**

☆ Functions where each x-value has at most one y-value AND each y-value has at most one x-value ☆

Given a graph of a function, is it one-to-one?

we apply the Horizontal Line Test

- if it passes → it is one-to-one
- if it fails → not one-to-one

**Exponential and Logarithmic Form:**

- Exponential has variables in powers/exponents
- Logarithmic has "log( )" or "ln( )"

← a logarithm must apply to something, kinda like a $\sqrt{\ }$ ← there must be something inside

Converting: ☆ the base always stays the base ☆

$b^x = a$ $\qquad \log_b(a) = x$ $\qquad \log_d(f) = g$

b is base $\qquad$ d is base

$\log_b(a) = x$

$b^x = a$ $\qquad d^g = f$



$a^x \cdot a^y \Leftrightarrow a^{x+y}$

**Properties of Logarithms:** Unfortunately, just comes down to memorization

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$\log_a(x) + \log_a(y) \Leftrightarrow \log_a(x \cdot y)$

$\log_a(x) - \log_a(y) \Leftrightarrow \log_a\left(\frac{x}{y}\right)$ ← came from negative log

$b \cdot \log_a(x) \Longleftrightarrow \log_a(x^b)$

☆ $\log_b(1) = 0$ ☆ ☆ $\log_b(b^x) = x$ ☆

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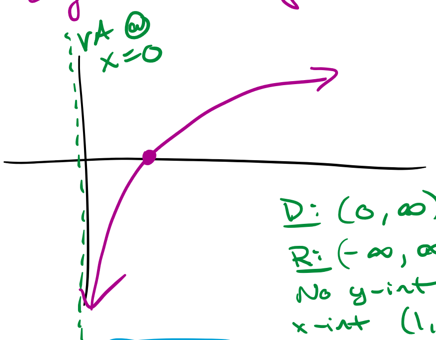
Logarithmic Function: Know properties of parent function

Logarithmic growth $\log_b(x)$, $b > 1$




D: $(0, \infty)$
R: $(-\infty, \infty)$
No y-int
x-int $(1, 0)$

Logarithmic Decay $\log_b(x)$, $0 < b < 1$

D: $(0, \infty)$
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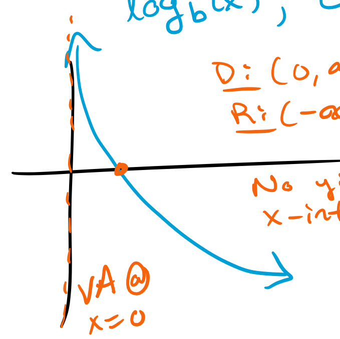



No y-int
x-int $(1, 0)$

Solving Exponential and Logarithmic Equations:

Exponential: we generally use logs (specifically "ln") unless we can rewrite all as the same base

Logarithmic: we generally use exponentials unless we can rewrite all as the same base

Exponential Models:

At this point, $A = Pe^{rt}$ is the only one you must memorize

$A = P\left(1 + \frac{r}{m}\right)^{mt}$ (pretty sure given if used; ask your prof)

$A$ = future value
$P$ = principle/starting amount
$t$ = time in years (unless specified otherwise)

$m$ = # time compounded
"quarterly" → $m = 4$
"weekly" → $m = 52$
⋮

3

**Examples:**

1. For the given functions, state the parent function. Then, state the transformations performed on the parent function to obtain the given function.

   (a) $g(x) = 5\sqrt[3]{x+1} + 2$

   parent function: $\sqrt[3]{x} = f(x)$

   ① shift left/right
   ②/③ vertical expansion/compression
   reflection across x-axis
   ④ shift up/down

   - shift left one unit ($x+1$ inside $\sqrt[3]{\ }$)
   - vertical expansion by factor of 5 (5 multiplied in front)
   - shift up two units (+2 at end)

   (b) $h(x) = -\frac{2}{3} \cdot \left(\frac{1}{7}\right)^{x-4}$

   parent function: $f(x) = \left(\frac{1}{7}\right)^x$

   - shift right four units ($x-4$ in power)
   - reflection across x-axis (negative coefficient)
   - vertical compression by a factor of $\frac{3}{2}$ (coefficient of $\frac{2}{3}$)

   reciprocal b/c "compression"

2. For the parent function $f(x) = x^2$, write a function $\boxed{k(x)}$ which is $f(x)$ with the following transformations:

   ④ • shift down 9 units
   2/3 • vertically $\boxed{\text{compress}}$ by a factor of 3 reciprocal
   ① • shift left 2 units
   2/3 • reflect across the $x$-axis

   $$k(x) = -\frac{1}{3}(x+2)^2 - 9$$

3. For the given functions, compute the indicated value, if it exists..

$$f(x) = 5x - x^2 \qquad g(x) = \frac{3}{x} - 7 \qquad h(x) = 2\sqrt{10 - x}$$

(a) $(f + g)(2) = f(2) + g(2)$

$= 6 + (-11/2)$

$= \frac{12}{2} - \frac{11}{2} = \boxed{\frac{1}{2}}$

$f(2) = 5(2) - (2)^2 = 10 - 4 = \boxed{6}$

$g(2) = \frac{3}{2} - 7 = \frac{3}{2} - \frac{14}{2} = \boxed{-\frac{11}{2}}$

y-values of f(x) & g(x)

(b) $\left(\frac{f}{h}\right)(-6) = \frac{f(-6)}{h(-6)}$

$= \frac{-66}{8} = \boxed{\frac{-33}{4}}$

$f(-6) = 5(-6) - (-6)^2$

$= -30 - 36 = -66$

$h(-6) = 2\sqrt{10-(-6)} = 2\sqrt{16}$

$= 8$

(c) $(g \circ f)(5) \rightarrow g(f(5))$

$f(5) = 5(5) - (5)^2 = 25 - 25 = 0$

$g(0)$

$g(0) = \frac{3}{0} - 7$

$\boxed{\text{undefined/DNE}}$

(d) $(h \circ f)(2) \rightarrow h(f(2))$

$f(2) = 6$

$h(6) = 2\sqrt{10-(6)} = 2\sqrt{4} = \boxed{4}$

★ base stays the base ★

4. Convert the following from an exponential equation to a logarithmic equation.

(a) $7^{3x} = 11$

$\log_7(11) = 3x$

(b) $\left(\frac{5}{2}\right)^2 = x$

$\log_{(5/2)}(x) = 2$

5. Convert the following from a logarithmic equation to an exponential equation.

(a) $\log_9(x+2) = -4$

$9^{-4} = x+2$

★ $\ln(x) = \log_e(x)$ ★

(b) $\ln(5) = x - 1$

$\log_e(5) = x - 1$

$e^{x-1} = 5$

6. Rewrite the following expression as a single logarithmic term.

$7\log_5(x) + 2 - \log_5(x-3) - 2\log_5(x-1)$

not a log!

"condensing logs"

① All logs & have same base

② Move all coefficients to powers

$2 \to \log_5(5^2)$

$7\log_5(x) + \log_5(25) - \log_5(x-3) - 2\log_5(x-1)$

$\underbrace{\log_5(x^7) + \log_5(25) - \log_5(x-3) - \log_5(x-1)^2}$

$\log_a(x) + \log_a(y) \to \log_a(x \cdot y)$

$\log_a(x) - \log_a(y) \to \log_a(x/y)$

- if it is inside a positive log → multiplied on top
- if it is inside a negative log → multiplied on bottom

$\log_5\left(\frac{x^7 \cdot 25}{(x-3)\cdot(x-1)^2}\right) \to \boxed{\log_5\left(\frac{25x^7}{(x-3)(x-1)^2}\right)}$

7. Fully expand and simplify the given logarithmic expression.

$$\log_3\left(\frac{27x^7z^4}{w^5y^{11}}\right)$$

← starts in top → positive log

← starts in bottom → negative log

$$\log_3(27) + \log_3(x^7) + \log_3(z^4) - \log_3(w^5) - \log_3(y^{11})$$

can I rewrite this in terms of the base of the log?

are there powers we can bring down?

$$\cancel{\log_3}(3^3) + 7\log_3(x) + 4\log_3(z) - 5\log_3(w) - 11\log_3(y)$$

$$\boxed{3 + 7\log_3(x) + 4\log_3(z) - 5\log_3(w) - 11\log_3(y)}$$

8. Determine the domain of the given functions in interval notation.

← so much of this on Exam III

3 domain restrictions:
① denominators ($\neq 0$)
② even roots (inside $\geq 0$)
③ logs (inside $> 0$)

(a) $f(x) = \dfrac{\sqrt{11-2x}}{\sqrt[3]{x-1}+2}$

even root: $11 - 2x \geq 0$

$$\frac{-2x}{-2} \leq \frac{-11}{-2}$$

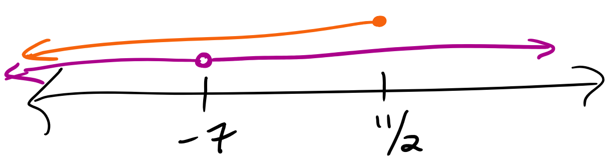
$$x \leq \frac{11}{2}$$

denom: $\sqrt[3]{x-1} + 2 \neq 0$

$$\left(\sqrt[3]{x-1}\right)^3 \neq (-2)^3$$

$$x - 1 \neq -8$$

$$x \neq -7$$




$$\boxed{D: (-\infty, -7) \cup (-7, 11/2]}$$

(b) $g(x) = \dfrac{x+5}{x-5} + 3\log_2(4x+20)$

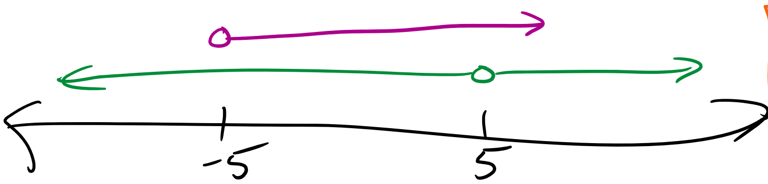
denom: $x - 5 \neq 0$

$$x \neq 5$$

log: $4x + 20 > 0$

$$\frac{4x}{4} > \frac{-20}{4}$$

$$x > -5$$




$$\boxed{D: (-5, 5) \cup (5, \infty)}$$

9. Solve the given equations for $x$. Express your answer in exact form and in terms of the natural logarithm when necessary.

no calculator

(a) $3^{2x} = 7^{x+1}$

different bases, so we can't just drop them & must use "ln"

$$\ln(3^{2x}) = \ln(7^{x+1})$$

★ this is b/c we want to move the x in the powers out of the powers

$\log_a(b^{x}) \rightarrow x \cdot \log_a(b)$

$$2x \cdot \ln(3) = (x+1) \cdot \ln(7)$$

$$2x \cdot \ln(3) = x \cdot \ln(7) + \ln(7)$$

$$2(x)\ln(3) - (x)\ln(7) = \ln(7)$$

cannot combine these → "factor"

$$\frac{x\,\cancel{(2\ln(3) - \ln(7))}}{\cancel{(2\ln(3) - \ln(7))}} = \frac{\ln(7)}{(2\ln(3) - \ln(7))}$$

$$\boxed{x = \frac{\ln(7)}{2\ln(3) - \ln(7)}}$$

(b) $7(2e^{8x} - 3) = -11$

★ a single exponential, we isolate that first

$$7(2 \cdot e^{8x} - 3) = -11$$

$$14 \cdot e^{8x} - 21 = -11$$

$$\frac{14e^{8x}}{14} = \frac{10}{14}$$

$$e^{8x} = \frac{5}{7}$$

★ when you isolate the exponential the other side must be positive

• if it isn't (negative or zero) then "no solution"

$$\cancel{\ln}(e^{8x}) = \ln(5/7)$$

$$\frac{8x}{8} = \frac{\ln(5/7)}{8}$$

$$\boxed{x = \frac{\ln(5/7)}{8} = \frac{1}{8}\ln(5/7)}$$

(c) $2^{4x} + 3 \cdot 2^{2x} = 10$

• two exponential terms, same base
• power on one is double the power on the other

(1) rewrite the one w/ bigger as the one w/ smaller power squared

(2) replace the exponential term w/ a single variable (that isn't being used)

(3) solve like quadratic equation

$$(2^{2x})^2 + 3 \cdot 2^{2x} = 10$$

$$y = 2^{2x}$$

$$y^2 + 3y = 10$$

$$y^2 + 3y - 10 = 0$$

$$(y+5)(y-2) = 0$$

$$y + 5 = 0 \qquad y - 2 = 0$$

$$2^{2x} + 5 = 0 \qquad 2^{2x} - 2 = 0$$

$2^{2x} + 5 = 0$

$2^{2x} = -5$

no solution

$\ln(2^{2x}) = \cancel{\ln(-5)}$

$2^{2x} - 2 = 0$

$2^{2x} = 2^1$

bases match, so drop

$2x = 1$

$$\boxed{x = \frac{1}{2}}$$

10. Solve the given equations for $x$. Express your answer in exact form.

(a) $\underbrace{\log(x) - \log(x-2)}_{\text{combine}} = 1$

★ For log equations, we want at most one log per side ★

• if there are constants, then we want no logs on that side

$\log_{10}\left(\frac{x}{x-2}\right) = 1$

$10^1 = \frac{x}{x-2}$

$(x-2)(10) = \left(\frac{x}{x-2}\right)(x-2)$

$10x - 20 = x$

$9x = 20$

$\boxed{x = \frac{20}{9}}$

★ ALWAYS check your answers w/ log equations

$\log(20/9)$ ✓

$\log(20/9 - 2)$ ✓

(b) $\underbrace{\ln(x+2) + \ln(x-3)}_{\text{combine}} = \underbrace{\ln(x) + \ln(x-5)}_{\text{combine}}$

$\ln((x+2)(x-3)) = \ln((x)(x-5))$ ★ log = log of same base, so drop logs

$(x+2)(x-3) = x(x-5)$

$\cancel{x^2} - x - 6 = \cancel{x^2} - 5x$

$4x - 6 = 0$

$4x = 6$

$\cancel{x = \frac{3}{2}}$ ★ Check This ★

$\boxed{\text{no solution}}$

$\ln\left(\frac{3}{2} + 2\right)$ ✓

$\ln\left(\frac{3}{2} - 3\right)$ ✗

(c) $\log_4(2x^2 + 2x) - \log_4(x+3) = \log_4(x+1)$

$\log_4\left(\frac{2x^2+2x}{x+3}\right) = \log_4(x+1)$ log = log

$\cancel{(x+3)}\left(\frac{2x^2+2x}{\cancel{x+3}}\right) = (x+1)(x+3)$

$2x^2 + 2x = x^2 + 4x + 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\boxed{x = 3}$, $\cancel{x = -1}$ check both

$x = 3$

$\log_4(2 \cdot 3^2 + 2 \cdot 3)$ ✓

$\log_4(3+3)$ ✓

$\log_4(3+1)$ ✓

$x = -1$

$\log_4(2 \cdot (-1)^2 + 2(-1))$

$2(1) - 2$

$= \cancel{0}$ must be positive

11. How much money would you need to deposit in a savings account that earns 4.5% annual interest compounded monthly if after 8 years, you want there to be $30,000 in the account?

$A = P\left(1 + \frac{r}{m}\right)^{mt}$

$A = 30{,}000$

$P = ?$

$r = .045$

$m = 12$

$t = 8$

$$30000 = P \cdot \left(1 + \frac{0.045}{12}\right)^{12(8)}$$

"exact answer" $\frac{30000}{(1.00375)^{96}} = \frac{P \cdot (1.00375)^{96}}{(1.00375)^{96}}$

★ Since P is money, round to two decimals ★

$\boxed{P = \$20{,}944.39}$ should be deposited

12. A savings account grows from an initial investment of $4,500 to $6,800 in 4 years. Calculate the annual interest rate for the savings account if the interest is compounded continuously. Express your answer in exact form then express your answer as a decimal rounded to 3 decimal places.

★ Must memorize: $A = Pe^{rt}$

$A = 6800$

$P = 4500$

$r = ?$

$t = 4$

$$\frac{6800}{4500} = \frac{4500e^{4r}}{4500}$$

$$\frac{68}{45} = e^{4r}$$

$$\ln\left(\frac{68}{45}\right) = \cancel{\ln}(\cancel{e}^{4r})$$

$$\frac{\ln\left(\frac{68}{45}\right)}{4} = \frac{4r}{4}$$

$$r = \underbrace{\frac{\ln\left(\frac{68}{45}\right)}{4}}_{\text{exact}} \approx 0.103$$

(if they asked for a %, our answer would be 10.3%)