Math 151
Week-In-Review 5
3.1 and 3.2
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$$\frac{d}{dx}\left[x^{100}\right] = 100x^{99}$$

Power Rule

$$\frac{d}{dx}\left[x^{n}\right] = n x^{n-1}$$

Problem Statements

1. If $f(x) = x^3$ show $f'(x) = 3x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = (x+h)(x^2+2xh+h^2)$$

$$x^3 + 2x^2h + xh^2 + hx^2 + 2xh^2 + h^3$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2 + 0 + 0 = \boxed{3x^2}$$

2. If $g(x) = c \cdot f(x)$ show $g'(x) = c \cdot f'(x)$.

$$\frac{d}{dx}\left[5 \cdot x^3\right] = 5 \cdot 3x^2 = 15x^2$$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}$$

$$= \lim_{h\to 0} c \cdot \left(\frac{f(x+h) - f(x)}{h}\right) = c \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \boxed{c \cdot f'(x)}$$

$$\frac{d}{dx}\left[e^{x}\right] = e^{x}$$

$\frac{d}{dx}[x^1] = 1 \cdot x^0 = 1$ $y = -5$

3. Calculate $f'(x)$ for the function $f(x) = x^{100} - 15x^5 + 6x - 5$.

$f'(x) = 100x^{99} - 15 \cdot 5x^4 + 6(1) - 0$

$$\boxed{f'(x) = 100x^{99} - 75x^4 + 6}$$

$\frac{d}{dx}[y] \downarrow$ $\frac{d^2}{dx^2}[y] \downarrow$

$f'(x) = \frac{dy}{dx} = y'$ $f''(x) = \frac{d^2y}{dx^2} = y''$

4. Find $\frac{d^2y}{dx^2}$ if $y = 5e^x - x^{7/8}$.

$\frac{7}{8} - \frac{8}{8} = \frac{-1}{8}$

$\frac{dy}{dx} = 5e^x - \frac{7}{8}x^{-1/8}$

$\frac{d^2y}{dx^2} = 5e^x - \frac{7}{8} \cdot \left(\frac{-1}{8}\right) x^{-9/8}$

$$\boxed{\frac{d^2y}{dx^2} = 5e^x + \frac{7}{64}x^{-9/8}}$$

$\frac{d}{dx}[e^x] = e^x \cdot \ln(e)$

$\frac{d}{dx}[5^x] = 5^x \cdot \ln(5)$

5. Determine the slope of the tangent line to the curve $g(x) = 3\sqrt{x} - \frac{1}{x^2} + 16$ when $x = 1$.

Derivative $\qquad g'(1) \qquad g(x) = 3x^{1/2} - x^{-2} + 16$

Start with $g'(x) = 3 \cdot \frac{1}{2}x^{-1/2} - (-2)x^{-3} + 0$

$$g'(x) = \frac{3}{2}x^{-1/2} + 2x^{-3}$$

$$g'(x) = \frac{3}{2\sqrt{x}} + \frac{2}{x^3}$$

$$g'(1) = \frac{3}{2\sqrt{(1)}} + \frac{2}{(1)^3} = \frac{3}{2} + 2 = \boxed{\frac{7}{2}}$$

6. Find the equation of the tangent line to the curve $f(x) = x^{3/2}$ that passes through the point $(0, -4)$.

$$f'(x) = \frac{3}{2}x^{1/2}$$

Both are Slope

$\sqrt{x} = 2$

$x = 4$

$$m = \frac{x^{3/2} - (-4)}{x - 0} = \frac{x^{3/2} + 4}{x}$$

$\frac{\Delta y}{\Delta x}$

$$f'(4) = \frac{3}{2}(4)^{1/2} = \frac{3}{2}(2) = 3$$

$$\boxed{y = 3x - 4}$$

$$\frac{3}{2}x^{1/2} = \frac{x^{3/2} + 4}{x}$$

$$\frac{3}{2}x^{3/2} = x^{3/2} + 4$$

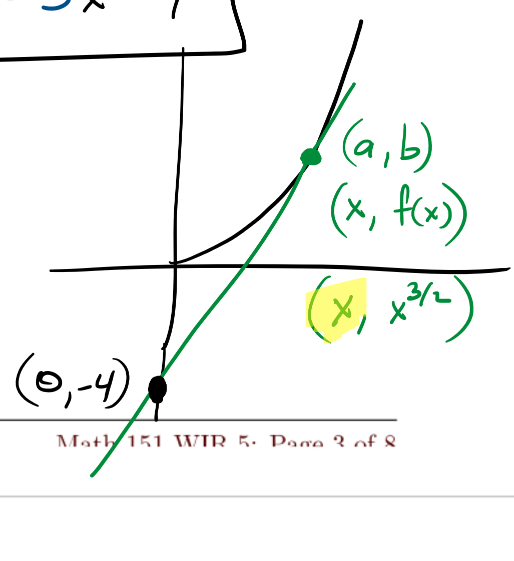
$-x^{3/2} \qquad -x^{3/2}$

$$\frac{1}{2}x^{3/2} = 4$$

$$x^{3/2} = 8$$

$$\sqrt{x^3} = 8$$

$$(\sqrt{x})^3 = 8$$



7. Evaluate $\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1}$.

$\frac{\Delta y}{\Delta x}$

$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

$\lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$f(x) = x^{1000}$

$a = 1$

$\frac{f(b) - f(a)}{b - a}$

$f(1) = 1^{1000} = 1$

Slope of tangent line to $f(x) = x^{1000}$ when $a = 1$.

$\frac{f(x) - f(a)}{x - a}$ ✓

$f'(x) = 1000x^{999}$

$f'(1) = 1000(1)^{999} = \boxed{1000}$

$\cancel{f'(x) = 1 \cdot e^x \,?}$

8. Consider the function $f(x) = \underbrace{(x-2)}_{1st}\underbrace{e^x}_{2nd}$.

Product Rule: $\frac{d}{dx}\left[\overset{1st}{f(x)} \cdot \overset{2nd}{g(x)}\right]$

$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$

$1st \; (2nd)' + 2nd \; (1st)'$

(a) Find $f'(x)$.

$$\boxed{f'(x) = (x-2)e^x + e^x(1)}$$

$$= e^x(x-2+1) = \boxed{\underbrace{(x-1)}_{1st}\underbrace{e^x}_{2nd}}$$

(b) Find $f''(x)$.

$$f''(x) = (x-1)e^x + e^x(1) = (x-1+1)e^x = \boxed{\underbrace{x}_{1st}\;\underbrace{e^x}_{2nd}}$$

(c) Find $f'''(x)$.

$$f'''(x) = x \cdot e^x + e^x(1) = \boxed{(x+1)e^x}$$

(d) Find a formula for the n^{th} derivative of the function, $f^{(n)}(x)$.

n	$f^{(n)}(x)$
0	$(x-2)e^x$
1	$(x-1)e^x$
2	xe^x
3	$(x+1)e^x$

$$\boxed{f^{(n)}(x) = (x-2+n)e^x}$$

Top

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Top, Bottom

Bot (Top)' − Top (Bot)'

(Bot)^2

9. Determine the equation of the tangent line to the curve $f(x) = \dfrac{4e^x - 2x^7}{x^2+1}$ when $x = 0$.

Point: $f(0) = \dfrac{4e^0 - 2(0)^7}{0^2+1} = \dfrac{4}{1} = 4 \quad (0,4)$

Slope:

$$f'(x) = \frac{(x^2+1)\left[4e^x - 14x^6\right] - (4e^x - 2x^7)(2x)}{(x^2+1)^2}$$

$$f'(0) = \frac{(0^2+1)\left[4e^0 - 14(0)\right] - \cancel{\left[4e^0 - 2(0)\right]2(0)}}{(0^2+1)^2}$$

0

$$= \frac{4}{1} = 4$$

Slope: $m = 4$

$$\boxed{\begin{aligned} y - 4 &= 4(x-0) \\ y &= 4x+4 \end{aligned}}$$

10. Suppose $p(x) = f(x)g(x)$ and $q(x) = \dfrac{f(x)}{g(x)}$, where f and g are differentiable functions. Use the following table to evaluate the derivatives below.

1st 2nd

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	−4	6	10
2	4	−7	5	6

(a) $p'(1)$

Start with $p'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

1st (2nd)' + 2nd (1st)'

$$p'(1) = f(1) \cdot g'(1) + g(1) \cdot f'(1)$$

$$= 3 \cdot 10 + (-4)(6)$$

$$= 30 + (-24) = \boxed{6}$$

(b) $q'(2)$

$q(x) = \dfrac{f(x)}{g(x)}$ Top / Bot

Bot (Top)' − Top (Bot)'

$$q'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$$

(Bot)^2

$$q'(2) = \frac{g(2)\, f'(2) - f(2)\, g'(2)}{[g(2)]^2} = \frac{(-7)(5) - (4)(6)}{(-7)^2}$$

$$= \frac{-35 - 24}{49} = \boxed{-\frac{59}{49}}$$

$\sqrt{x} = x^{1/2}$

11. If $f(x)$ is a differentiable function, find the derivative of $y = \dfrac{1 + x \cdot f(x)}{\sqrt{x}}$.

$$y' = \frac{\underbrace{(\sqrt{x})}_{\text{Bot}} \cdot \underbrace{\left[0 + \underbrace{x \cdot f'(x) + f(x)(1)}_{\text{Product Rule}}\right]}_{(\text{Top})'} - \underbrace{(1 + x \cdot f(x))}_{\text{Top}} \underbrace{\left(\frac{1}{2} x^{-1/2}\right)}_{(\text{Bot})'}}{\underbrace{(\sqrt{x})^2}_{(\text{Bot})^2}}$$

12. If $g(x)$ is a differentiable function, find the derivative of $y = \underbrace{(x \cdot e^x)}_{1^{st}} \cdot \underbrace{g(x)}_{2^{nd}}$.

$$y' = \underbrace{(x \cdot e^x)}_{1^{st}} \cdot \underbrace{g'(x)}_{(2^{nd})'} + \underbrace{g(x)}_{2^{nd}} \cdot \underbrace{\left[\underbrace{x \cdot e^x + e^x \cdot (1)}_{\text{Product Rule}}\right]}_{(1^{st})'}$$

$$y' = x e^x \cdot g'(x) + x \cdot e^x \cdot g(x) + (1) e^x g(x)$$

$$\frac{d}{dx}\left[10\cdot x^2\right] = 10\cdot(2x) + \cancel{x^2(0)}$$

$$\frac{d}{dx}\left[10x^2\cdot e^x\right] = 10x^2\cdot e^x + e^x\cdot(20x)$$

$$\frac{d}{dx}\left[\frac{x^2}{5}\right] = \frac{d}{dx}\left[\frac{1}{5}\cdot x^2\right]$$