CHARACTERIZING DIFFERENTIAL EQUATIONS

Review

- The **order** of a differential equation is the order of the highest derivative.
- Ordinary vs partial differential equations
  - A **ordinary** differential equation has derivatives with respect to one variable.
  - A **partial** differential equation has derivatives with respect to more than one variable.
- Linear ODEs
  - A **linear** ODE has the form

    $$a_n(x)y^{(n)}(x) + ... + a_1(x)y'(x) + a_0(x)y(x) = g(x).$$

    Said another way, it satisfies the following conditions:
    * All the y's are in different terms.
    * None of the y's are inside a function or to a power.
    * The y's can be multiplied by a function of x.
    * There can be terms that depend only on x.
- Homogeneous linear ODEs
  - A linear ODE is **homogeneous** if the $g(t)$ term is 0.
- Separable ODEs
  - An ODE is **separable** if you can write it in the form $y' = f(x)g(y)$.
- Autonomous ODEs
  - An ODE is **autonomous** if the dependent variable (x) does not show up explicitly. i.e., if x does not show up outside of y.

Exercise 1

Classify the following differential equations. In particular, put it into one (or more) of the following categories and state the order.

- Partial differential equation
- Ordinary differential equation
  - Separable
  - Linear
    - Homogeneous
  - Autonomous

1. $y^2 - y'' + 6 = 0$

   2nd order linear and autonomous ODE

2. $f_x - f_y = xf$

   1st order PDE

3. $y'(x) + x^2y(x) = 3y(x)$ $= y' + (x^2-3)y = 0$

   1st order homogeneous linear ODE

4. $g' = x^2 \sin(g)$

   1st order separable ODE

5. $\sin(x)w''' + w - 3 = 0$

   3nd order linear ODE

6. $u''(x) = \sin(u(x))$

   2nd order autonomous ODE

7. $f^{(5)} - \cos(x^2)f''' - \tan(x)f = 3\tan(x)$

   5th order linear ODE

SOLVING DIFFERENTIAL EQUATIONS

Review

- First order ODEs
  - You do **NOT** need to guess which method to use to solve a 1st order ODE!
  - How to determine which method to use:
    1. Is the equation **separable**?
       If yes, use separation of variables.
    2. Is the equation **linear**?
       If yes, use the method of integrating factors.

    2'. Is it a Bernoulli equation[1]?
       If yes, then use $v = y^{1-n}$.

    3. Is the equation **exact**?
       If yes, then use the method for exact equations.

    3'. Is it a homogenous equation[2]?
       If yes, then use $v = y/x$ to get a separable equation.

    4. If none of the above, then try to find an integrating factor to make the equation exact.[3]
- Second order linear ODEs
  - Homogeneous with constant coefficients
    1. Look for solutions of the form $y(t) = e^{rt}$.
    2. Find the characteristic equation.
    3. Find the roots of the characteristic equation.
    4. The general solution is given by
       * Distinct real roots: $c_1e^{r_1t} + c_2e^{r_2t}$
       * Complex roots: $c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$
       * Repeated real roots: $c_1e^{rt} + c_2te^{rt}$
    5. If you have initial conditions, use them to solve for c_1 and c_2.
  - Nonhomogeneous
    * Method of undetermined coefficients (if constant coefficients and you can guess)
    * Variation of parameters

[1] A Bernoulli equation has the form $y' + p(t)y = q(t)y^n$. Not all instructors cover this. You can find examples of Bernoulli equations in Section 2.4 of the textbook, #23–25.

[2] This is NOT the same as the homogeneous linear equations that are covered in Chapter 3. The terminology is confusing. "Homogeneous equation" here refers to a 1st order ODE that can be written in the form $y' = f(\frac{y}{x})$. Not all instructors cover this. You can find examples of these in Section 2.2 of the textbook, #25–31.

[3] Not all instructors cover making an equation exact by using an integrating factor.

Exercise 2

Find the general solution to

$$t^2y' + ty - t = 0.$$

$$\mu y' + \underbrace{\frac{1}{t}\mu}_{\frac{d\mu}{dt} = \frac{1}{t}\mu} y = \frac{1}{t}\mu$$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{t}\,dt$$

$$\ln|\mu| = \ln|t|$$

$$\mu = t$$

$$\frac{d}{dt}(ty) = \frac{1}{t}t = 1$$

$$ty(t) = t + c$$

$$\boxed{y(t) = 1 + \frac{c}{t}}$$

Exercise 3

Solve the initial value problem

$$u' - tu^{-2} = 0, \quad u(1) = -1.$$

$$\frac{du}{dt} = \frac{t}{u^2}$$

$$\int u^2\,du = \int t\,dt$$

$$\frac{1}{3}u^3 = \frac{1}{2}t^2 + c$$

$$u^3 = \frac{3}{2}t^2 + c$$

$u(1) = 3$:

$$(-1)^3 = \frac{3}{2}(1)^2 + c$$

$$-1 = 3/2 + c$$

$$c = -5/2$$

$$u^3 = \frac{3}{2}t^2 - 5/2$$

$$\boxed{u(t) = \left(\frac{3}{2}t^2 - \frac{5}{2}\right)^{1/3}}$$

Exercise 4

Find the general solution to

$$f'' = 3f' - 2f.$$

$$f'' - 3f' + 2f = 0$$

$$r^2 - 3r + 2 = 0$$

$$(r-1)(r-2) = 0$$

$$r = 1, 2$$

$$\boxed{f(t) = c_1 e^{t} + c_2 e^{2t}}$$

Exercise 5

Find the general solution to

$$w'' + 4w' + 4w = 5e^t.$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2$$

$$w_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Guess: $w_p(t) = Ae^t$

$$Ae^t + 4Ae^t + 4Ae^t = 5e^t$$

$$9Ae^t = 5e^t$$

$$9A = 5$$

$$A = 5/9$$

$$\boxed{w(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{5}{9} e^t}$$

Exercise 6

Find the general solution to

$$(4x - 2y)y' + 4y = -2x.$$

$$\underbrace{2x + 4y}_{M} + \underbrace{(4x - 2y)}_{N} y' = 0$$

$$M_y = 4 = N_x = 4 \quad \checkmark \text{exact}$$

$$\psi_x = 2x + 4y \qquad \psi = x^2 + 4xy + c(y)$$

$$\Rightarrow$$

$$\psi_y = 4x - 2y \qquad \psi = 4xy - y^2 + c(x)$$

$$\psi(x,y) = \boxed{x^2 + 4xy - y^2 = c}$$

Exercise 7

Find the general solution to

$$3g'' - 2g' + 4g = 0.$$

$$3r^2 - 2r + 4 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(4)(3)}}{2(3)} = \frac{1}{3} \pm \frac{\sqrt{-44}}{6}$$

$$= \frac{1}{3} \pm i\frac{\sqrt{11}}{3}$$

$$\boxed{g(t) = c_1 e^{t/3} \cos\left(\frac{\sqrt{11}}{3} t\right) + c_2 e^{t/3} \sin\left(\frac{\sqrt{11}}{3} t\right)}$$

Exercise 8

Solve the initial value problem

$$f = -\frac{1}{9}f'', \quad f(0) = -2, \quad f'(0) = 1.$$

$\frac{1}{9}f'' + f = 0$

$\frac{1}{9}r^2 + 1 = 0$

$r^2 = -9$

$r = \pm 3i$

$f(t) = c_1 \cos(3t) + c_2 \sin(3t)$

$f'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t)$

$f(0) = c_1 = -2$

$f'(0) = 3c_2 = 1 \Rightarrow c_2 = 1/3$

$$\boxed{f(t) = -2\cos(3t) + \frac{1}{3}\sin(3t)}$$

Exercise 9

Find a that makes the equation exact.

$$\underbrace{x^3 + y^a}_{M} + \underbrace{2xy}_{N}y' = 0.$$

want

$M_y = ay^{a-1} = N_x = 2y$

$ay^{a-1} = 2y$

$\Rightarrow \boxed{a = 2}$

Exercise 10

Solve by first finding an integrating factor that makes the equation exact.

$$\underbrace{y}_{M} + \underbrace{(2xy - e^{-2y})}_{N}y' = 0.$$

μ depends only on x:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

$$= \frac{1-2y}{\underbrace{2xy - e^{-2y}}_{\text{cannot depend on } y}}\mu$$

μ depends only on y:

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$$

$$= \frac{2y-1}{\underbrace{y}_{\text{does not depend on } x}}\mu \quad \checkmark$$

$$\int \frac{d\mu}{\mu} = \int \frac{2y-1}{y}\,dy$$

$$\ln|\mu| = \int\left(2 - \frac{1}{y}\right)dy$$

$$\ln|\mu| = 2y - \ln|y|$$

$$\mu(y) = e^{2y - \ln|y|} = e^{2y}e^{\ln|y^{-1}|}$$

$$= y^{-1}e^{2y} \leftarrow \text{integrating factor}$$

$$y\mu + \mu \cdot (2xy - e^{-2y})y' = 0$$

$$e^{2y} + y^{-1}e^{2y}(2xy - e^{-2y})y' = 0 \quad (y \neq 0)$$

$$\underbrace{e^{2y}}_{\psi_x} + \underbrace{(2xe^{2y} - y^{-1})}_{\psi_y} y' = 0$$

$$\psi_x = e^{2y} \qquad \psi = xe^{2y} + c(y)$$

$$\psi_y = 2xe^{2y} - y^{-1} \quad \Rightarrow \quad \psi = xe^{2y} - \ln|y| + c(x)$$

$$\psi(x,y) = \boxed{xe^{2y} - \ln|y| = c \quad \text{or} \quad y = 0}$$

Case $y = 0$:

$$\frac{dy}{dx} = 0$$

plug into diff eq:

$$0 + (2x \cdot 0 - e^{-0}) \cdot 0 = 0 \checkmark$$

So, $y = 0$ is also a solution.

Exercise 11

Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 5y' + 6y = 4e^{-2t} + 3t^3.$$

What is an appropriate guess for the particular solution y_p?

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$r = 2, 3$

$y_h(t) = c_1 e^{2t} + c_2 e^{3t}$

$\boxed{y_p(t) = Ae^{-2t} + Bt^3 + Ct^2 + Dt + E}$

Exercise 12

Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 2y' + y = 3e^t - t\sin(t).$$

What is an appropriate guess for the particular solution y_p?

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$y_h(t) = c_1 e^t + c_2 te^t$

$\boxed{y_p(t) = At^2e^t + (Bt+C)(D\sin(t) + E\cos(t))}$

Exercise 13

Given that $\underbrace{x^2}_{y_1}$ and $\underbrace{x^{-1}}_{y_2}$ are solutions to the corresponding homogeneous equation, find a particular solution to

$$x^2y'' - 2y = 3x^2 - 1, \quad x > 0.$$

put into standard form first!

$$y'' - 2x^{-2}y = \underbrace{3 - x^{-2}}_{g}$$

$$W[x^2, x^{-1}](x) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -1 - 2 = -3$$

$$y_p(t) = -x^2\int \frac{x^{-1}(3-x^{-2})}{-3}\,dx + x^{-1}\int \frac{x^2(3-x^{-2})}{-3}\,dx$$

$$= \frac{1}{3}x^2\int (3x^{-1} - x^{-3})\,dx - \frac{1}{3}x^{-1}\int (3x^2 - 1)\,dx$$

$$= \frac{1}{3}x^2\left(3\ln(x) + \frac{1}{2}x^{-2}\right) - \frac{1}{3}x^{-1}(x^3 - x)$$

$$\boxed{= x^2\ln(x) + \frac{1}{6} - \frac{1}{3}x^2 + \frac{1}{3}}$$

ANALYSIS OF ODES

Review

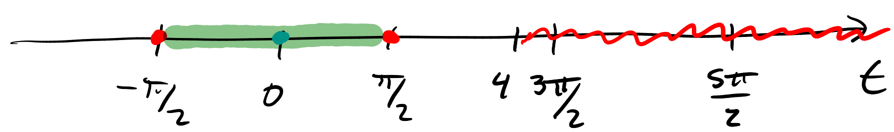
- Where is a solution valid?
  - Solution is valid on a **single interval** where the solution is a function that is defined and differentiable.
- Existence and uniqueness
  - **1st order linear ODEs:** If p and g are continuous on an interval $I = (a, b)$ containing the initial condition t_0, then the initial value problem
  $$y' + p(t)y = g(t), \qquad y(t_0) = y_0$$
  has a unique solution on I.
  - **1st order nonlinear ODEs:** Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $(a, b) \times (c, d)$ containing the point (t_0, y_0). Then, there is a unique solution to the initial value problem
  $$y' = f(t, y), \qquad y(t_0) = y_0$$
  on a sufficiently small interval $I_h = (t_0 - h, t_0 + h)$ around t_0.
  - **2nd order linear ODEs:** Consider the initial value problem
  $$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$
  If p, q, and g are continuous on an open interval $I = (a, b)$ that contains the point t_0, then there is exactly one solution to the initial value problem and the solution exists throughout the entire interval I.
- The **Wronskian** of y_1 and y_2 is defined by
$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$
- $\{y_1, y_2\}$ is a **fundamental set of solutions** means that the general solution is $c_1y_1 + c_2y_2$.
- Slope fields
- Equilibrium solutions
- Stability of equilibrium solutions
  - **(Asymptotically) stable:** If you start near it, you go in towards it.
  - **Unstable:** If you start near it, you go away from it.
  - **Semistable:** If you start near on one side, you go towards it, but if you start near on the other side, you go away from it.
- Phase line diagrams

Exercise 14

Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$y' - \underbrace{t^2 \tan(t)}_{t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots} y = \underbrace{\sqrt{4-t}}_{4-t \geq 0}, \quad y(0) = \pi.$$

$t \leq 4$



Unique solution to the IVP exists on $(-\pi/2, \pi/2)$.

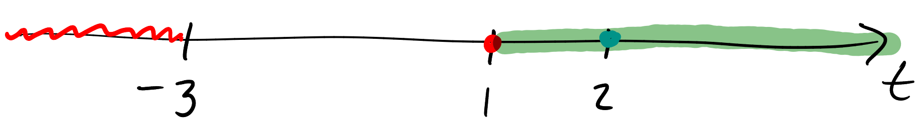
Exercise 15

Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$(t-1)w'' + w' - \ln(t+3)w = t^3\cos(t), \quad w(2) = -2 \quad w'(2) = 7.$$

$$w'' + \underbrace{\frac{1}{t-1}}_{t \neq 1} w' - \underbrace{\frac{\ln(t+3)}{t-1}}_{t \neq 1,\ t+3>0} w = \underbrace{\frac{t^3\cos(t)}{t-1}}_{t \neq 1}$$

$t > -3$



Unique solution exists on $(1, \infty)$.

Exercise 16

For which values t_0 and y_0 is the following initial value problem guaranteed to have a unique solution?

$$t^2y^2 - (t+y)y' = 0, \quad y(t_0) = y_0.$$

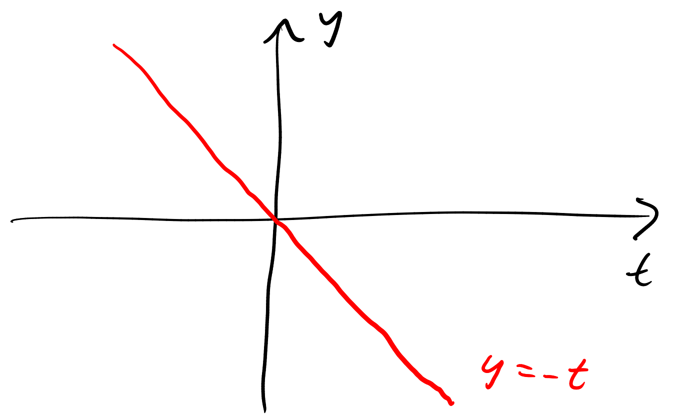
$$(t+y)y' = t^2y^2$$

$$y' = \frac{t^2y^2}{t+y} = f(t,y)$$

$t+y \neq 0$

$$\frac{\partial f}{\partial y} = \frac{(t+y)2t^2y - t^2y^2(1)}{(t+y)^2}$$

$t+y \neq 0$



There is a unique solution as long as $y_0 \neq -t_0$.

Exercise 17

Show that x and xe^x form a fundamental set of solutions to

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0.$$

First, show they are solutions!

$y_1 = x$
$y_1' = 1$
$y_1'' = 0$

$$x^2 \cdot 0 - x(x+2)(1) + (x+2)x = 0$$
$$0 = 0 \checkmark$$

$y_2 = xe^x$
$y_2' = xe^x + e^x$
$y_2'' = xe^x + 2e^x$

$$x^2(x\cancel{e^x} + 2\cancel{e^x}) - x(x+2)(x\cancel{e^x} + \cancel{e^x}) + (x+2)x\cancel{e^x} = 0$$
$$x^3 + 2x^2 - x(x^2+3x+2) + x^2 + 2x = 0$$
$$\cancel{x^3} + \cancel{2x^2} - \cancel{x^3} - \cancel{3x^2} - \cancel{2x} + \cancel{x^2} + \cancel{2x} = 0 \checkmark$$

Check Wronskian:

$$W[x, xe^x](1) = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{vmatrix} = \begin{vmatrix} 1 & e \\ 1 & 2e \end{vmatrix} = 2e - e = e \neq 0$$

So, $\{x, xe^x\}$ is a fundamental set of solutions.

Exercise 18

Solve for the explicit solution $u(x)$. Where is the solution to the initial value problem valid? How does this depend on a?

$$u' = u^2, \quad u(0) = a.$$

$$\int \frac{du}{u^2} = \int dx \qquad (u \neq 0)$$

$$-u^{-1} = x + c$$

solve for c:

$$-a^{-1} = 0 + c$$

$$c = \frac{-1}{a}$$

solve for u:

$$\boxed{\begin{array}{ll} u(x) = \dfrac{-1}{x - \frac{1}{a}} & \text{if } a \neq 0. \\[2ex] u(x) = 0 & \text{if } a = 0. \end{array}}$$

Case $u = 0$:

$$\frac{du}{dx} = 0$$

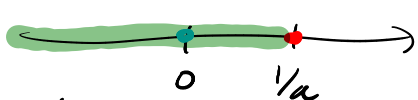
plug into diff eq:

$$0 = 0^2 \checkmark$$

$u(x) = 0$ is a solution

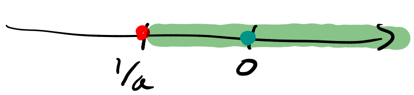
$a = 0$: valid for $-\infty < x < \infty$

$a > 0$:



valid for $-\infty < x < \frac{1}{a}$

$a < 0$:



valid for $\frac{1}{a} < x < \infty$

Exercise 19

Consider the differential equation

$$f' = f(f-2)^2(f-4)$$

(a) Find the equilibrium solutions

(b) Draw the phase line diagram

(c) Sketch the slope field

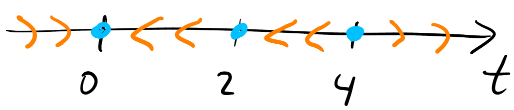
(d) Determine the stability of each equilibrium solution

(e) Determine $\lim_{t\to\infty} f(t)$ for different initial values $f(0)$.

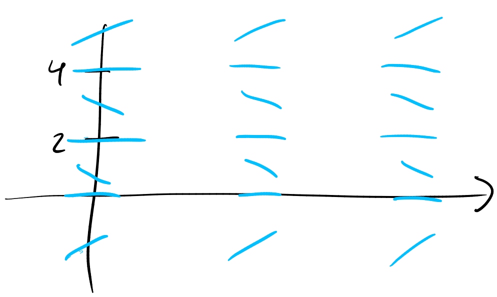
(a) $f' = f(f-2)^2(f-4) = 0$

$$\boxed{f = 0,\ 2,\ 4}$$

(b)



(c)



(d) 0 is stable, 2 is semistable, 4 is unstable

(e) If $f(0) < 2$, then $f(t) \to 0$.

If $2 \le f(0) < 4$, then $f(t) \to 2$.

If $f(0) = 4$, then $f(t) \to 4$.

If $f(0) > 4$, then $f(t) \to \infty$.