3.2: SOLUTIONS OF LINEAR HOMOGENEOUS ODES

Review

- **Existence and uniqueness:** Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

If p, q, and g are continuous on an open interval $I = (a, b)$ that contains the point t_0, then there is exactly one solution to the initial value problem and the solution exists throughout the entire interval I.

- **Principle of superposition:** If y_1 and y_2 are solutions to a homogeneous ODE, then

$c_1 y_1 + c_2 y_2$ is also a solution.

- A set of functions is called a **fundamental set of solutions** if adding them together with constants forms the general solution.

$\{y_1, y_2\}$ is a fundamental set of solutions $\iff$ $c_1 y_1 + c_2 y_2$ is the general solution

- The **Wronskian** of y_1 and y_2 is defined as

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

- Interpretation of the Wronskian:

If $W[y_1, y_2](t) = 0$, then $\{y_1, y_2\}$ is <u>not</u> a fundamental set of solutions.

If $W[y_1, y_2](t) \neq 0$, then $\{y_1, y_2\}$ is a fundamental set of solutions.

- The Wronskian only needs to be checked at a single value of t in the interval where the solution exists.

Exercise 1

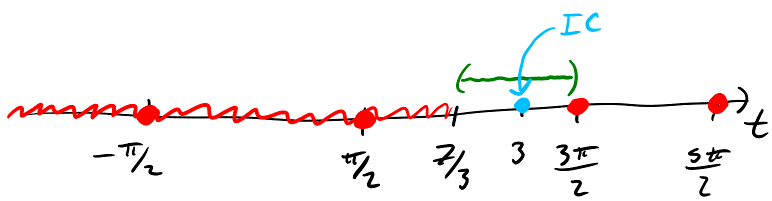
Is the following initial value problem guaranteed to have a unique solution? If so, on what interval is it guaranteed to exist?

$$y'' - \underbrace{\sec(t)}_{\frac{1}{\cos(t)}}y' + (t^2+1)y = \underbrace{\sqrt{3t-7}}_{\text{need } 3t-7\geq 0}, \qquad y(3) = -3, \qquad y'(3) = 2.$$

need $\cos(t) \neq 0$

$\cos(t) = 0$ when $t = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$3t - 7 \geq 0 \Rightarrow 3t \geq 7 \Rightarrow t \geq \frac{7}{3}$



IVP has a unique solution on $\left(-\frac{7}{3}, \frac{3\pi}{2}\right)$.

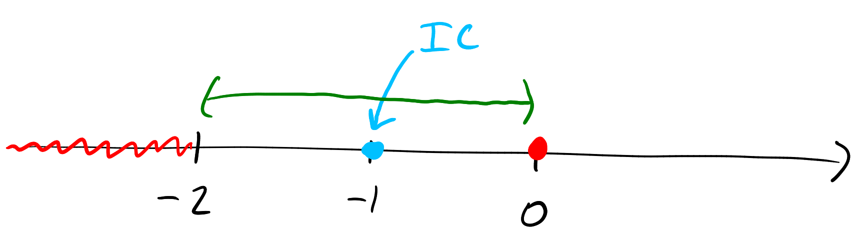
Exercise 2

Is the following initial value problem guaranteed to have a unique solution? If so, on what interval is it guaranteed to exist?

$$tf'' + \sin(t)f' + \ln(t+2)f = 8, \qquad f(-1) = 7, \qquad f'(-1) = -4.$$

Write into standard form: $f'' + \underbrace{\frac{\sin(t)}{t}}_{t\neq 0} f' + \underbrace{\frac{\ln(t+2)}{t}}_{t\neq 0,\ t+2>0} = \underbrace{\frac{8}{t}}_{t \neq 0}.$

$t > -2$



IVP has a unique solution on $(-2, 0)$.

Exercise 3

Do $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for the following differential equation?

$$y'' - 3y' + 3y = 0.$$

First, check if they are even solutions.

$y_1 = e^t$
$y_1' = e^t$
$y_1'' = e^t$

$$\Rightarrow \quad e^t - 3e^t + 3e^t = 0$$

$$\cancel{e^t = 0}$$

e^t is not even a solution.

So, no, they are not a fundamental set of solutions.

Exercise 4

Do $y_1(t) = e^t$ and $y_2(t) = t+1$ form a fundamental set of solutions for the following differential equation?

$$ty'' - (t+1)y' + y = 0, \qquad t < 0.$$

Check if they are solutions:

$$y_1: \ \cancel{te^t} - (\cancel{t}+1)e^t + \cancel{e^t} = 0 \ \checkmark$$

$$y_2: \ t(0) - \cancel{(t+1)}(1) + \cancel{(t+1)} = 0 \ \checkmark$$

Check the Wronskian:

$$W[y_1, y_2](-1) = \begin{vmatrix} y_1(-1) & y_2(-1) \\ y_1'(-1) & y_2'(-1) \end{vmatrix} = \begin{vmatrix} e^{-1} & 0 \\ e^{-1} & 1 \end{vmatrix} = e^{-1} \neq 0$$

Yes, they form a fundamental set of solutions.

Exercise 5

Do $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t + \pi)$ form a fundamental set of solutions to the following differential equation?

$$y'' + y = 0.$$

First, are they solutions?

$y_1 = \cos(t)$

$y_1' = -\sin(t)$

$y_1'' = -\cos(t)$

$\Rightarrow -\cos(t) + \cos(t) = 0$ ✓

$y_2 = \sin(t - \pi/2)$

$y_2' = \cos(t - \pi/2)$

$y_2'' = -\sin(t - \pi/2)$

$\Rightarrow -\sin(t - \pi/2) + \sin(t - \pi/2) = 0$ ✓

Check the Wronskian:

$$W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} \cos(0) & \sin(-\pi/2) \\ -\sin(0) & \cos(-\pi/2) \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

Since the Wronskian is 0, these are <u>not</u> a fundamental set of solutions.

Exercise 6

Show that $y(t) = c_1 t + c_2 t \ln(t)$ is the general solution to the differential equation,

$$t^2 y'' - ty' + y = 0, \qquad t > 0.$$

$y = c_1 t + c_2 t \ln(t)$ is the general solution $\iff$ $\{t, t\ln(t)\}$ are a fundamental set of solutions

$y_1(t) = t$ $\qquad$ $y_2(t) = t\ln(t)$

$y_1'(t) = 1$ $\qquad$ $y_2'(t) = \ln(t) + t\frac{1}{t} = \ln(t) + 1$

$y_1''(t) = 0$ $\qquad$ $y_2''(t) = \frac{1}{t}$

Are they solutions?

$y_1: \; t^2(0) \cancel{- t(1)} \cancel{+ t} = 0$ ✓

$y_2: \; \cancel{t^2\left(\frac{1}{t}\right)} - t\left(\cancel{\ln(t)} + \cancel{1}\right) + \cancel{t\ln(t)} = 0$ ✓

Check the Wronskian:

$$W[y_1, y_2](1) = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$$

Therefore, $\{t, t\ln(t)\}$ is a fundamental set of solutions.

$\Rightarrow$ $y = c_1 t + c_2 t\ln(t)$ is the general solution.

3.3 & 3.4: SECOND ORDER LINEAR ODES

Review

- A **second order linear ODE with constant coefficients** has the form

$$ay'' + by' + cy = g(t).$$

- It is **homogeneous** if $g(t) = 0$.
- Process for **solving** a second order homogeneous linear ODE:
  1. Look for solutions of the form $y(t) = e^{rt}$.
  2. Find the characteristic equation.
  3. Find the roots of the characteristic equation.
  4. The general solution is given by
     - Distinct real roots: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
     - Complex roots: $y(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt)$ $(r = a \pm ib)$
     - Repeated real roots: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$
  5. If you have initial conditions, use them to solve for c_1 and c_2.

Exercise 7

Find the general solution to the differential equation

$$y'' + 10y' + 25y = 0.$$

Plug in $y = e^{rt}$:

$$r^2 \cancel{e^{rt}} + 10r\cancel{e^{rt}} + 25\cancel{e^{rt}} = 0$$

$$r^2 + 10r + 25 = 0 \leftarrow \text{characteristic equation}$$

$$(r+5)^2 = 0$$

$$r = -5$$

Plug into the repeated roots equation:

$$\boxed{y(t) = c_1 e^{-5t} + c_2 t e^{-5t}}$$

Exercise 8

Find the general solution to the differential equation

$$y'' - 9y' + 20y = 0.$$

Characteristic equation:

$$r^2 - 9r + 20 = 0$$

$$(r-4)(r-5) = 0$$

$$r = 4, 5$$

Plug into the distinct roots equation:

$$\boxed{y(t) = c_1 e^{4t} + c_2 e^{5t}}$$

Exercise 9

Solve the initial value problem

$$f'' - 2f' + 8f = 0, \quad f(0) = 0, \quad f'(0) = 1.$$

Characteristic equation:

$$r^2 - 2r + 8 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(8)}}{2} = 1 \pm \frac{\sqrt{-28}}{2} = 1 \pm i\sqrt{7}$$

General solution:

$$f(t) = c_1 e^t \cos(\sqrt{7}\,t) + c_2 e^t \sin(\sqrt{7}\,t)$$

Use the initial conditions to solve for c_1 and c_2:

$$f(0) = c_1 \cancelto{1}{e^0} \cancelto{1}{\cos(0)} + c_2 e^0 \cancelto{0}{\sin(0)} = c_1 = 0$$

$$f(t) = c_2 e^t \sin(\sqrt{7}\,t)$$

$$f'(t) = c_2 e^t \sin(\sqrt{t}\,7) + \sqrt{7}\, c_2 e^t \cos(\sqrt{7}\,t)$$

$$f'(0) = c_2 e^0 \cancelto{0}{\sin(0)} + \sqrt{7}\, c_2 \cancelto{1}{e^0} \cancelto{1}{\cos(0)} = \sqrt{7}\, c_2 = 1$$

$$\Rightarrow c_2 = \frac{1}{\sqrt{7}}$$

$$\boxed{f(t) = \frac{1}{\sqrt{7}} e^t \sin(\sqrt{7}\,t)}$$

Exercise 10

Find the general solution to the differential equation

$$4g'' + g = 0.$$

Characteristic equation:

$$4r^2 + 1 = 0$$

$$4r^2 = -1$$

$$r^2 = \frac{-1}{4}$$

$$r = \pm \frac{1}{2} i$$

General solution:

$$\boxed{g(t) = c_1 \sin\left(\tfrac{1}{2}t\right) + c_2 \cos\left(\tfrac{1}{2}t\right)}$$

Exercise 11

Find the general solution to the differential equation

$$3y'' - 2y' - y = 0$$

Characteristic equation:

$$3r^2 - 2r - 1 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(-1)(3)}}{2(3)} = \frac{2 \pm \sqrt{16}}{6} = \frac{2 \pm 4}{6} = \frac{-1}{3}, 1$$

General solution:

$$\boxed{y(t) = c_1 e^{\frac{-1}{3}t} + c_2 e^{t}}$$

Exercise 12

Solve the initial value problem

$$f'' - 4f' + 4f = 0, \qquad f(0) = 2, \qquad f'(0) = -1.$$

Characteristic equation:

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2$$

General solution:

$$f(t) = c_1 e^{2t} + c_2 t e^{2t}$$

Use ICs to solve for c_1 and c_2:

$$f'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$$

$$f(0) = c_1 e^0 + 0 = c_1 = 2$$

$$f'(0) = 2c_1 e^0 + c_2 e^0 + 0 = 2c_1 + c_2 = -1$$

$$\Rightarrow 2(2) + c_2 = -1$$

$$c_2 = -5$$

$$\boxed{f(t) = 2e^{2t} - 5te^{2t}}$$