7.7: MATRIX EXPONENTIALS

Review

- How to **diagonalize** a 2×2 matrix A
  1. Find the eigenvalues λ_1 and λ_2 and eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
  2. $A = PDP^{-1}$, where
$$P = \left[\ \boldsymbol{\xi}^{(1)} \;\middle|\; \boldsymbol{\xi}^{(2)}\ \right], \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$
- **Matrix exponential**
$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1}.$$
- The matrix exponential is useful for solving the initial value problem
$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$
  In particular, the solution is
$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

Exercise 1

Solve the initial value problem by using the matrix exponential.

$$\mathbf{x}' = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

7.8: REPEATED EIGENVALUES

Review

- **How to solve** a homogeneous linear system with constant coefficients, $\mathbf{x}' = A\mathbf{x}$.
  1. Assume your solution has the form $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$.
  2. Plug this in to get an eigenvalue problem.
  3. Solve for the eigenvalues.
  4. Based on the eigenvalues:
     - Real distinct eigenvalues:
       (a) Solve for the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
       (b) General solution is $c_1 e^{r_1 t}\boldsymbol{\xi}^{(1)} + c_2 e^{r_2 t}\boldsymbol{\xi}^{(2)}$.
     - Complex eigenvalues:
       (a) Solve for one eigenvector $\boldsymbol{\xi}$.
       (b) Find the real and imaginary parts of the solution $e^{(a+ib)t}\boldsymbol{\xi}$.
       (c) General solution is c_1(real part) $+ c_2$(imaginary part).
     - Repeated eigenvalues:
       (a) Solve for the eigenvector(s).
       (b) If there are two independent eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$:
           (i) General solution is $c_1 e^{rt}\boldsymbol{\xi}^{(1)} + c_2 e^{rt}\boldsymbol{\xi}^{(2)}$.
       (c) If there is only one independent eigenvector $\boldsymbol{\xi}$:
           (i) Solve for the generalize eigenvector $\boldsymbol{\eta}$.
           (ii) General solution is $c_1 e^{rt}\boldsymbol{\xi} + c_2\left(te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}\right)$.
- The **generalize eigenvector** η can be found via the equation

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi},$$

  where r is the eigenvalue and $\boldsymbol{\xi}$ is the eigenvector.

Exercise 2

Find the general solution and sketch the phase portrait.

$$\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$$

Exercise 3

Find the general solution and sketch the phase portrait.

$$\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

7.9: NONHOMOGENEOUS LINEAR SYSTEMS

Review

- A nonhomogeneous linear system has the form

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

- There are 4 methods for solving these:
  1. Method of undetermined coefficients
     - Works if $P(t) = A$ and you can guess the particular solution.
  2. Variation of parameters
     - Fundamental matrix: $\Psi(t) = \left[\ \mathbf{x}^{(1)} \ \middle|\ \cdots \ \middle|\ \mathbf{x}^{(n)}\ \right]$.
     - $\mathbf{x}_p(t) = \Psi(t) \int \Psi^{-1}(t)\mathbf{g}(t)\,\mathrm{d}t$.
     - Always works.
  3. Laplace transform
     - Works if $P(t) = A$ and you can take the Laplace transform of everything.
  4. Diagonalization
     - Works if the matrix is diagonalizable.

Exercise 4

Find the general solution using the method of undetermined coefficients.

$$\mathbf{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix}$$

Exercise 5

Consider the system of differential equations

$$\mathbf{x}' = \frac{1}{t}\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3t \end{bmatrix}.$$

The general solution to the homogeneous system is

$$\mathbf{x}_h(t) = c_1 t^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 t^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find a particular solution to the nonhomogeneous system using variation of parameters.

Exercise 6

Solve the initial value problem using the Laplace transform. (Stop when you get to $X(s)$.)

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \cos(t) \\ t^3 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Exercise 7

Find the general solution using diagonalization.

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3t \\ t \end{bmatrix}$$