Math 152 - Week-In-Review 6

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i) linear terms.
ii) quadratic terms.
iii) repeating vs non repeating.
iv) min 2 reduced terms.

1. Write out the form of the following partial decompisition (PFD) fractions. Do not solve.

(a) $\frac{1-x}{x^3+x^4}$ $= \frac{1-x}{x^3(1+x)} = \frac{1-x}{x\cdot x\cdot x\cdot(1+x)}$

$= \frac{A}{1+x} + \frac{B}{x} + \frac{C}{x\cdot x} + \frac{D}{x\cdot x\cdot x}$

$= \frac{A}{1+x} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$

(b) $\frac{x}{x^2+x-6}$ $= \frac{x}{(x+3)(x-2)}$

$= \frac{A}{x+3} + \frac{B}{x-2}$

(c) $\frac{1+16x}{(2x-3)(x+5)^2(x^2+4)}$ $= \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)(x+5)} + \frac{Dx+E}{x^2+4}$

$(2x-3)$: linear (non repeting)

$(x+5)^2$: $(x+5)(x+5)$ linear (repeating)

(x^2+4): sum of squares quadratic (non repeting)

Evaluate the following integrals:

2. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \frac{A}{2x+1}\,dx + \int \frac{B}{x-1}\,dx$

lin lin

$$(2x+1)(x-1)\left[\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}\right]$$

$$5x+1 = A\underbrace{(x-1)} + B\underbrace{(2x+1)}$$

@ $x=1$ $\quad 6 = A(0) + B(3) \longrightarrow B = \frac{6}{3} = 2$

@ $x=0$ $\quad 1 = A(-1) + B(1)$

$$1 = -A + 2 \longrightarrow A = 1$$

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \frac{1}{\underbrace{2x+1}}\,dx + \int \frac{2}{x-1}\,dx$$

$2\int \frac{dx}{x-1}$

$u = 2x+1$

$du = 2dx$ $= \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u|$

$$\boxed{\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C}$$

3. $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx \quad = \int_0^1 \frac{A}{(x+1)}dx + \int_0^1 \frac{B}{(x+1)(x+1)}dx + \int_0^1 \frac{C}{(x+2)}dx$

$(x+1)(x+1)(x+2)$

$$x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)(x+1)$$

@ $x=-1 \Rightarrow +1-1+1 = 1 = A(0) + B(1) + C(0) \longrightarrow B=1$

@ $x=-2 \Rightarrow 4-2+1 = 3 = A(0) + B(0) + C(-1)(-1) \longrightarrow C=3$

@ $x=0 \Rightarrow 1 = A(1)(2) + \cancel{B}^{1}(2) + \cancel{C}^{3}(1)(1)$

$1 = 2A + 2 + 3 \longrightarrow A=-2$

$$\int_0^1 \frac{-2}{(x+1)}dx + \int_0^1 \frac{1}{(x+1)^2}dx + \int_0^1 \frac{3}{(x+2)}dx$$

$\int \frac{1}{u^2}du = -\frac{1}{u}$

$-2\ln|x+1|\Big|_0^1 \quad + \quad -\frac{1}{(x+1)}\Big|_0^1 \quad 3\ln|x+2|\Big|_0^1$

$\underbrace{-2\left[\ln(2)-\cancel{\ln(1)}^{0}\right]}_{-2\ln(2)} \quad \underbrace{-1\left[\frac{1}{2}-\frac{1}{1}\right]}_{+\frac{1}{2}} \quad \underbrace{3\left[\ln(3)-\ln(2)\right]}_{+3\left(\underbrace{\ln 3-\ln 2}_{\ln\frac{3}{2}}\right)}$

$$\boxed{\text{Ans: } -2\ln(2)+\frac{1}{2}+3\ln\left(\frac{3}{2}\right)}$$

$a^2-b^2=(a+b)(a-b)$

a^2+b^2 is non-reducible → sum of squares.

4. $\int \frac{x^2}{x^4-81}\,dx$ $= \int \frac{x^2}{(x^2)^2-(9)^2}dx = \int \frac{x^2\,dx}{(x^2+9)(x^2-9)}$

$= \int \frac{x^2\,dx}{(x^2+9)(x+3)(x-3)}$

$$\frac{x^2}{(x-3)(x+3)(x^2+9)} = \frac{A}{x-3} + \frac{B}{x+3} + \frac{Cx+D}{x^2+9}$$

(x^2-9)

$$x^2 = A(x+3)(x^2+9) + B(x-3)(x^2+9) + (Cx+D)(x+3)(x-3)$$

$$= A(x^3+3x^2+9x+27) + B(x^3-3x^2+9x-27)$$

$$= Ax^3 + 3Ax^2 + 9Ax + 27A + Bx^3 - 3Bx^2 + 9Bx - 27B$$
$$Cx^3 + Dx^2 - 9Cx - 9D$$

$$x^3(0)+x^2(1)+x(0)+(0) = x^3(A+B+C) + x^2(3A-3B+D) + x(9A+9B-9C) + (27A-27B-9D)$$

✓ $A+B+C=0$

$3A-3B+D=1$

$9(A+B-C)=0$ ← ✓ $9A+9B-9C=0$

$\boxed{27A-27B-9D=0}$

✓ $3A-3B-D=0$

$3A-3B=D$

$-3B-3B=D$

$D=-6B$

$A+B+C=0$
$A+B-C=0$
$2A+2B=0$

$A+B=0$

$A=-B$

$3A-3B+D=1$

$-3B-3B-6B=1$

$-12B=1$

$\boxed{B=-1/12}$

$\boxed{A=1/12}$

$\boxed{D=-6\cdot -\frac{1}{12}=\frac{1}{2}}$

$C=?$

$A+B+C=0$

$C=-A-B$

$=-\frac{1}{12}+\frac{1}{12}=0$

$\boxed{C=0}$

$$\int \frac{(1/12)}{x-3}dx + \int \frac{(-1/12)}{x+3}dx + \int \frac{(1/2)}{x^2+9}dx$$

$$\boxed{\frac{1}{12}\ln|x-3| - \frac{1}{12}\ln|x+3| + \frac{1}{2}\left(\frac{1}{3}\right)\arctan\left(\frac{x}{3}\right) + C}$$

5. $\displaystyle\int \frac{x^5 - x^4 - 2x^2 + 2x + 5}{x^4 + x^3}\,dx$ $\sim \frac{x^5}{x^4}$ $\longrightarrow$ long division

$$\begin{array}{r l}
 & x - 2 \\
x^4+x^3 \,\big)\! & x^5 - x^4 + 0x^3 - 2x^2 + 2x + 5 \\
 & -(x^5 + x^4) \\
 & -2x^4 + 0x^3 - 2x^2 + 2x + 5 \\
 & -(-2x^4 - 2x^3) \\
 & 2x^3 - 2x^2 + 2x + 5
\end{array}$$

$= (x-2) + \dfrac{2x^3-2x^2+2x+5}{x^4+x^3}$

Ex: $\frac{15}{4} = 3 + \frac{3}{4}$

$$\boxed{\int \frac{x^5-x^4-2x^2+2x+5}{x^4+x^3}\,dx = \underbrace{\int (x-2)\,dx}_{\frac{x^2}{2}-2x} + \int \frac{2x^3-2x^2+2x+5}{\underbrace{x^4+x^3}_{x\cdot x\cdot x\,(x+1)}}\,dx}$$

$$\frac{2x^3-2x^2+2x+5}{x\cdot x\cdot x\cdot(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{1+x}$$

$$2x^3-2x^2+2x+5 = A\,(\cancel{x})\overset{x^2}{(x)}(1+x) + B\,(x)(1+x) + C(1+x) + Dx^3$$

$$= A(x^2+x^3) + B(x+x^2) + C + Cx + Dx^3$$

$$x^3(2) + x^2(-2) + x(2) + (5) = x^3(A+D) + x^2(A+B) + x(B+C) + (C)$$

$\overset{1}{\cancel{A}}+D=2$ $\quad \boxed{D=1}$

$\boxed{C=5}$

$B+\overset{5}{\cancel{C}}=2$ $\quad \boxed{B=-3}$

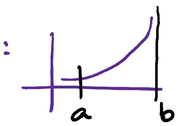
$A+\overset{-3}{\cancel{B}}=-2$ $\quad \boxed{A=1}$

$$= \int \frac{1}{x}\,dx + \int \frac{-3}{x^2}\,dx + \int \frac{5}{x^3}\,dx + \int \frac{1}{1+x}\,dx$$

$$\ln|x| - 3\left(-\frac{1}{x}\right) + \frac{5}{-2x^2} + \ln|1+x| + C$$

$$\boxed{\frac{x^2}{2} - 2x + \ln|x| + \frac{3}{x} - \frac{5}{2x^2} + \ln|1+x| + C}$$

Improper Integrals → Type I : ∞ in bound(s) of integration

Type II : 

→ check Domain

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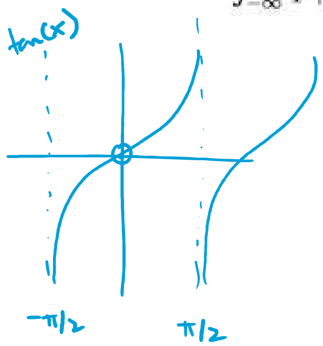
Compute the following integrals or show that they diverge.

Type I.

6. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$



$$= \underbrace{\int_{-\infty}^{0} \frac{dx}{1+x^2}}_{} + \underbrace{\int_{0}^{\infty} \frac{dx}{1+x^2}}_{}$$

$$\lim_{t\to(-\infty)} \arctan(x)\Big|_t^0 + \lim_{t\to\infty} \arctan(x)\Big|_0^t$$

$$\underbrace{\arctan(0)}_{0} - \underbrace{\arctan(-\infty)}_{-\pi/2} + \underbrace{\arctan(\infty)}_{\pi/2} - \underbrace{\arctan(0)}_{0}$$

$$+\frac{\pi}{2} + \frac{\pi}{2}$$

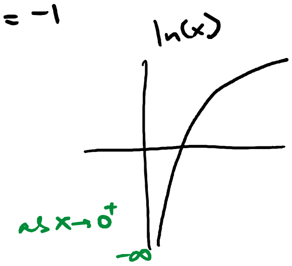
$$= \pi \quad \text{Ans.}$$

7. $\displaystyle\int_{0}^{\infty} \frac{dx}{(x+2)(x+3)}$ $\xrightarrow{\text{PFD}}$ $\displaystyle\int \frac{A}{x+2}\,dx + \int \frac{B}{x+3}\,dx$

Important
$\infty - \infty$: indeterminate difference
$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

If you solve for A and B, you get $A = 1$ and $B = -1$

$$= \int_0^{\infty} \frac{1}{x+2}\,dx + \int_0^{\infty} \frac{-1}{x+3}\,dx$$



$$\lim_{t\to\infty} \Big[\ln|x+2| - \ln|x+3| \Big|_0^t \Big]$$

$$= \underbrace{\Big[\ln(\infty) - \ln(\infty)\Big]}_{\infty - \infty} - \underbrace{\Big[\ln(2) - \ln(3)\Big]}_{-\ln\left(\frac{2}{3}\right)}$$

UB LB

$$\lim_{x\to\infty} \ln\left|\frac{x+2}{x+3}\right| = \ln\left[\lim_{x\to\infty}\underbrace{\left(\frac{x+2}{x+3}\right)}_{\frac{\infty}{\infty}}\right] = \ln(1) = 0$$

$$\frac{\infty}{\infty} \xrightarrow{\text{L'H}} \lim_{x\to\infty}\left(\frac{1}{1}\right) = 1$$

Ans.
$0 - \ln\left(\frac{2}{3}\right)$
or
$+\ln\left(\frac{3}{2}\right)$

8. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,dx \xrightarrow{\text{IBP}}$ $\quad u = \ln x \quad dv = \frac{1}{x^3}\,dx$

$\quad du = \frac{1}{x}\,dx \quad v = -\frac{1}{2x^2}$

$$= \ln x\left(-\frac{1}{2x^2}\right) - \int \left(-\frac{1}{2x^2}\right)\cdot\frac{1}{x}\,dx$$

$$= -\frac{1}{2}\cdot\frac{\ln x}{x^2} + \frac{1}{2}\int \frac{1}{x^3}\,dx$$

$$= -\frac{1}{2}\frac{\ln x}{x^2} - \frac{1}{4}\cdot\frac{1}{x^2}\Bigg|_1^{\infty}$$

$-\frac{1}{2}\left(\frac{\infty}{\infty}\right) \to \text{L'H} \to 0$

$$-\frac{1}{2}\lim_{x\to\infty}\frac{\ln x}{x^2} \sim \frac{\infty}{\infty}$$

$$-\frac{1}{2} \xrightarrow{\text{L'H}} \lim_{x\to\infty}\frac{1/x}{2x} = -\frac{1}{2}\lim_{x\to\infty}\left(\cancel{\frac{1}{2x^2}}^{\,0}\right)$$

$$= \left[\underbrace{-\frac{1}{2}\frac{\ln(\infty)}{(\infty)^2}} - \underbrace{\cancel{\frac{1}{4}\cdot\frac{1}{(\infty)^2}}^{\,0}}\right] - \left[\underbrace{-\frac{1}{2}\frac{\cancel{\ln(1)}^{\,0}}{1^2}}_{0} - \underbrace{\frac{1}{4}\cdot\frac{1}{1^2}}_{-\frac{1}{4}}\right].$$

Ans: $0 - 0 - 0 + \frac{1}{4}$

$\frac{1}{4}$

$\tan\left(\frac{\pi}{2}\right) = \infty$

9. $\displaystyle\int_{\pi/4}^{\pi/2} \tan^2 x\,dx$

$$= \int_{\pi/4}^{\pi/2} (\sec^2 x - 1)\,dx$$

$$= \tan(x) - x\Bigg|_{\pi/4}^{\pi/2}$$

$$= \left[\cancel{\tan\left(\frac{\pi}{2}\right)}^{\,\infty} - \frac{\pi}{2}\right] - \left[\cancel{\tan\left(\frac{\pi}{4}\right)}^{\,1} - \frac{\pi}{4}\right].$$

$\Rightarrow \infty$

Ans: Integral diverges.

10. $\int_2^{10} \frac{dx}{x^2-9}$

why is it improper?

Domain of $\frac{1}{x^2-9}$: division by zero @ $x = \pm 3$

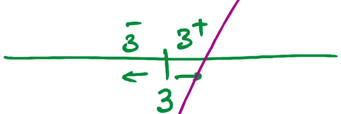
$2 < x=3 < 10$

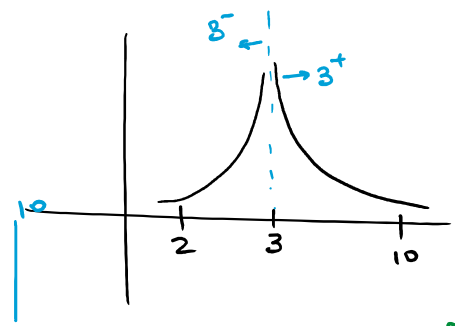
$$\int_2^{3^-} \frac{dx}{x^2-9} + \int_{3^+}^{10} \frac{dx}{x^2-9}$$

LB asymptote: (+)ve limit

UB asymptote: (-)ve limit

PFD : $\frac{A}{x+3} + \frac{B}{x-3}$





$$= -\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| \Big|_{3^+}^{10}$$

$$= \left[-\frac{1}{6}\ln|10+3| + \frac{1}{6}\ln|10-3|\right] - \left[-\frac{1}{6}\ln|3^++3| + \underbrace{\frac{1}{6}\ln|\cancel{3^+-3}|}_{\substack{-\infty \\ \frac{1}{6}\ln(0^+)}}\right]$$

$$-\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| \Big|_2^{3^-}$$

$$\left[-\frac{1}{6}\ln|3^-+3| + \underbrace{\frac{1}{6}\ln|3^--3|}_{\frac{1}{6}\ln|0^-|}\right] - \left[-\frac{1}{6}\ln 5 + \cancel{\frac{1}{6}\ln|-1|}\right]$$

$\to (-\infty)$ $\therefore$ Integral diverges.

$= \frac{1}{6}\ln(0^+)$

$\to -\infty$

$\therefore$ Integral diverges.

11. $\displaystyle\int_1^{\infty} \sin(\pi x)\, dx$

$$= -\frac{\cos(\pi x)}{\pi}\Big|_1^{\infty}$$

$$= -\frac{1}{\pi}\left[\underbrace{\cos(\infty)}_{\text{Oscillates between } \pm 1} - \cos(\pi)\right].$$

$\therefore$ Integral diverges by oscillation

$$\boxed{-1 \leq \cos(x) \leq +1}$$

$$\boxed{-1 \leq \sin(x) \leq +1}$$

$$0 \leq \cos^2(x) \leq 1$$

$$0 \leq \sin^2(x) \leq 1$$

12. $\displaystyle\int_0^2 \frac{dx}{4x-5}$

Use the Comparison test to determine whether the following integrals converge or diverge.

13. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$

14. $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\, dx$

$-1 \le \cos(x) \le +1$

15. $\displaystyle\int_1^\infty \frac{2+\overbrace{\cos x}^{[-1,\,1]}}{\sqrt{x^4+x^2}}\,dx$

divergence　　　convergence

$$\frac{1}{\sqrt{x^4+x^2}} \le \underbrace{\frac{2+\cos(x)}{\sqrt{x^4+x^2}} \le \frac{3}{\sqrt{x^4+x^2}}}$$

$\lim\limits_{x\to\infty} \sqrt{x^4+x^2} \sim \sqrt{x^4}$

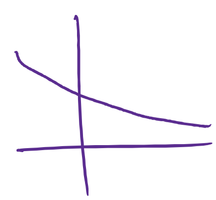
$$\int^\infty \frac{1}{\sqrt{x^4}}\,dx = \int_1^\infty \frac{1}{x^2}\,dx$$

converge by p-series.

Ans: $\displaystyle\int_1^\infty \frac{2+\cos(x)}{\sqrt{x^4+x^2}}\,dx$ will converge by comparison to $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx$

16. $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$

$\lim\limits_{x\to\infty} e^{-x} \to 0$



compare to $\displaystyle\int_1^\infty \frac{2}{x}\,dx \longrightarrow$ diverges.