Math 152 - Week-In-Review 1

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Let us start with some review of antiderivatives. Evaluate the following integrals:

1. $\int \left(x^5 + 6x + 1 + \frac{2}{x} + \frac{3}{x^3}\right) dx$ $3x^{-3}$

$= \frac{x^6}{6} + \frac{\overset{3}{\cancel{6}}x^2}{\cancel{2}} + x + 2\ln|x| + \frac{3x^{-2}}{-2} + C$

$-\frac{3}{2x^2}$

2. $\int \left(\sqrt{x} + \sqrt[3]{x^2} + x^{5/3} + \frac{1}{\sqrt{x}} + \frac{4}{x^{2/5}}\right) dx$ $= \int (x^{1/2} + x^{2/3} + x^{5/3} + x^{-1/2} + 4x^{-2/5}) dx$

$= \frac{2}{3}x^{3/2} + \frac{3}{5}x^{5/3} + \frac{3}{8}x^{8/3} + 2x^{1/2} + 4 \cdot \frac{5}{3}x^{3/5} + C$

3. $\int \left(e^x + \sin x + \sec^2 x + \csc x \cot x + \frac{10}{1+x^2}\right) dx$

$= e^x - \cos(x) + \tan(x) - \operatorname{cosec}(x) + 10\arctan(x) + C$

4. $\int_0^3 \frac{1}{9+x^2}\, dx$ $= \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_0^3$

$9 + x^2 = (3)^2 + x^2$

$= \frac{1}{3}\arctan(1) - \frac{1}{3}\cancel{\arctan(0)}^{0}$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

$= \frac{1}{3}\left(\frac{\pi}{4}\right)$

Use the substitution rule to evaluate the following integrals:

1. $\int x(8x^2+1)^{3/2}\,dx$

$u = 8x^2+1$

$du = 8(2x)\,dx$

$\frac{du}{16} = \underbrace{x\,dx}$ OR $\frac{du}{16x} = dx$

$= \int u^{3/2} \cdot \frac{du}{16}$

$= \frac{1}{\cancel{16}_{8}} \cdot \frac{\cancel{2}}{5} u^{5/2} + C$

$= \frac{1}{40}(8x^2+1)^{5/2} + C$

2. $\int \frac{5\cos\sqrt{x}}{\sqrt{x}}\,dx$ $= 5\int \cos(\sqrt{x}) \cdot \underbrace{\frac{1}{\sqrt{x}}\,dx}$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= 5\int \cos(u) \cdot 2\,du$

$= 10\sin(u) + C$

$= 10\sin(\sqrt{x}) + C$

3. $\int \frac{(\ln x)^3}{x}\,dx$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$= \int u^3\,du$

$= \frac{u^4}{4} + C$

$= \frac{1}{4}(\ln x)^4 + C$

4. $\int \frac{e^x}{e^x+1}\,dx$

$u = e^x + 1$

$du = e^x\,dx$

$= \int \frac{du}{u}$

$= \ln|u| + C$

$= \ln(e^x+1) + C$

5. $\int e^{5-7x}\,dx$

$= \dfrac{e^{5-7x}}{(-7)} + c$

$\int e^{x}\,dx = e^{x} + c$

$\int e^{2x}\,dx = \dfrac{e^{2x}}{2} + c$

$\int e^{5x}\,dx = \dfrac{e^{5x}}{5} + c$

$\int \cos(x)\,dx = \sin(x) + C$

$\int \cos(5x)\,dx = \dfrac{\sin(5x)}{5} + c$

6. $\displaystyle\int \frac{e^{3x}+5}{e^{3x}+15x}\,dx$

$u = e^{3x} + 15x$

$du = (3e^{3x} + 15)\,dx$

$du = 3(e^{3x}+5)\,dx$

$\dfrac{du}{3} = (e^{3x}+5)\,dx$

$= \displaystyle\int \frac{du/3}{u}$

$= \dfrac{1}{3}\displaystyle\int \frac{du}{u} = \frac{1}{3}\ln|u| + c$

$= \dfrac{1}{3}\ln|e^{3x} + 15x| + c$

when is $\ln|x|$ not necessary?

(1) $e^{x} + c$

(2) sum of squares

7. $\int (3\cos x + 6)\sec^2(\sin x + 2x)\,dx$

$3(\cos x + 2)$

$u = \sin x + 2x$

$du = (\cos x + 2)\,dx$

$= 3\displaystyle\int \sec^2(u)\,du$

$= 3\tan(u) + c = 3\tan(\sin x + 2x) + c$

8. $\displaystyle\int \frac{3\sin x\cos x}{1+\cos^2 x}\,dx$

$u = 1 + \cos^2(x)$

$\dfrac{du}{-2} = \dfrac{-2}{-2}\cos(x)\sin(x)\,dx$

$3\displaystyle\int \frac{du/-2}{u} = -\frac{3}{2}\int \frac{du}{u}$

$= -\dfrac{3}{2}\ln|u| + c = -\dfrac{3}{2}\ln(1+\cos^2 x) + c$

9. $\displaystyle\int_{e^3}^{e^5} \frac{1}{x\ln x}\,dx$ $= \displaystyle\int_{(LB)\, x=e^3}^{(UB)\, x=e^5} \frac{1}{\ln(x)} \cdot \underbrace{\frac{1}{x}\,dx}$

$\boxed{u=\ln(x)}$
$du = \frac{1}{x}\,dx$

UB: $x=e^5 \rightarrow u=\ln(e^5) = 5\ln e$, $u=5$
LB: $x=e^3 \rightarrow u=\ln(e^3)=3$

$= \displaystyle\int_{u=3}^{u=5} \frac{1}{u}\cdot du = \ln|u| \Big|_3^5$

$= \ln(5) - \ln(3) = \ln\left(\frac{5}{3}\right)$

10. $\displaystyle\int_1^2 \frac{(x+1)}{x(x+2)}\,dx$ $= \displaystyle\int_{x=1}^{x=2} \frac{(x+1)}{x^2+2x}\,dx$

$u = x^2+2x$
$du = (2x+2)\,dx$
$= 2(x+1)\,dx$
$\frac{du}{2} = (x+1)\,dx$

$= \displaystyle\int \frac{du/2}{u} = \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+2x| \Big|_{x=1}^{x=2}$

$= \frac{1}{2}\ln(8) - \frac{1}{2}\ln(3) = \boxed{\frac{1}{2}\ln\left(\frac{8}{3}\right)}$

11. $\displaystyle\int_{x=0}^{x=1} x(x+5)^9\,dx$

$u = x+5$
$du = dx$
$x = u-5$

UB: $x=1 \rightarrow u = 1+5=6$
LB: $x=0 \rightarrow u = 0+5=5$

$= \displaystyle\int_{u=5}^{u=6} (u-5)u^9\,du$

$= \displaystyle\int_5^6 (u^{10}-5u^9)\,du = \frac{u^{11}}{11} - \frac{5u^{10}}{10_2}\Big|_5^6$

$= \boxed{\left(\frac{6^{11}}{11} - \frac{6^{10}}{2}\right) - \left(\frac{5^{11}}{11} - \frac{5^{10}}{2}\right)}$

12. $\displaystyle\int_0^1 \frac{11+2x}{x^2+1}\,dx$

$= \displaystyle\int_0^1 \frac{11}{x^2+1}\,dx + \int_0^1 \frac{2x}{x^2+1}\,dx$

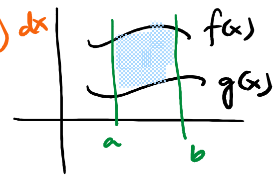
$u = x^2+1$
$du = 2x\,dx$

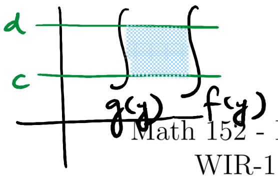
$\displaystyle\int \frac{du}{u} = \ln|u|$
$= \ln(x^2+1)$

$= 11\arctan(x) + \ln(x^2+1)\Big|_{x=0}^{x=1}$

$= 11\arctan(1)^{\pi/4} + \ln(2) - [11\arctan(0)^{0} + \ln(1)^{0}] = \boxed{11\left(\frac{\pi}{4}\right) + \ln(2)}$

$\int_a^b (T-B)\,dx$

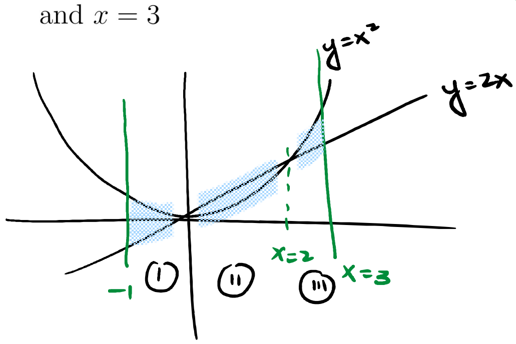




$\int_c^d (R-L)\,dy$

Finding the area between curves:

13. Find the area enclosed by the parabola $y = x^2$ and the line $y = 2x$ between $x = -1$ and $x = 3$



check for intersection point(s)

$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

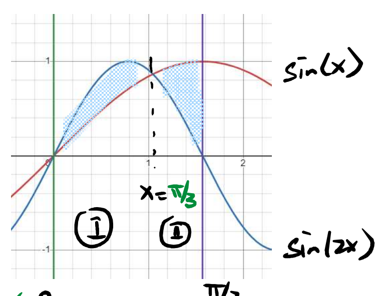
$x=0 \qquad x=2$

$$A = \int_{-1}^{0} (x^2-2x)\,dx + \int_0^2 (2x-x^2)\,dx + \int_2^3 (x^2-2x)\,dx$$

$$= \frac{x^3}{3} - x^2 \Big|_{-1}^{0} + \left(x^2 - \frac{x^3}{3}\right)\Big|_0^2 + \left(\frac{x^3}{3} - x^2\right)\Big|_2^3$$

$$= \underbrace{0 - \left(-\tfrac{1}{3} - 1\right)}_{+\frac{4}{3}} + \underbrace{\left(4 - \tfrac{8}{3} - 0\right)}_{+\frac{4}{3}} + \underbrace{\left(\tfrac{27}{3} - 9 - \left(\tfrac{8}{3} - 4\right)\right)}_{+\frac{4}{3}} = 3\left(\tfrac{4}{3}\right) = 4 \text{ Ans}$$

14. Find the area bounded by the curves $y = \sin x$, $y = \sin 2x$ and $0 \le x \le \pi/2$.



	0	π/4	π/2
sin(x)	0	√2/2	1
sin(2x)	0	sin(2·π/4) = 1	sin(2·π/2) = 0

intersection: $\sin(x) = \sin(2x)$

$\sin(x) = 2\sin(x)\cos(x)$

$\sin(x) - 2\sin(x)\cos(x) = 0$

$\sin(x)(1 - 2\cos(x)) = 0.$

$x = 0 \qquad 1 - 2\cos(x) = 0$

$1 = 2\cos(x)$

$\frac{1}{2} = \cos(x)$

$x = \pi/3$

$$\boxed{A = \int_0^{\pi/3} (\sin(2x) - \sin(x))\,dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin(2x))\,dx = \tfrac{1}{2}}$$

One way to do this: $\int \sin(2x)\,dx = \int 2\sin(x)\cos(x)\,dx$

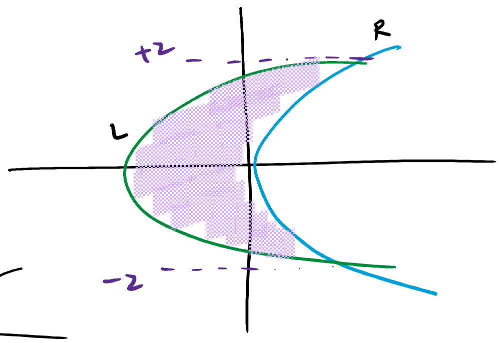
$u = \sin(x)$

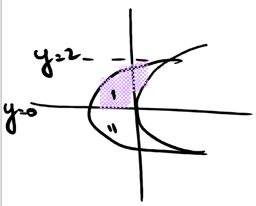
$du = \cos(x)\,dx$

$= 2\int u\,du = \frac{\cancel{2}u^2}{\cancel{2}} = \sin^2(x)$

both are symmetric about x-axis

15. Find the area bounded by the curves $x = y^2$ and $x = 2y^2 - 4$.





Intersection point(s)

$y^2 = 2y^2 - 4$

$y^2 - 4 = 0$

$y = \pm 2$

$$A = \int_{-2}^{2} [y^2 - (2y^2 - 4)]\,dy = \int_{-2}^{2} \underbrace{[y^2 - (2y^2-4)]}_{4-y^2}\,dy$$

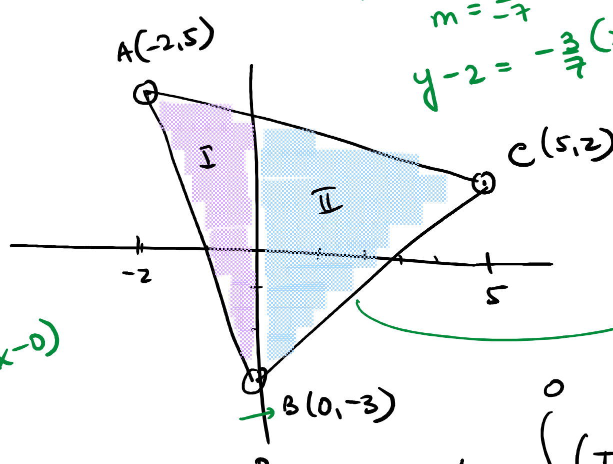
I can use symmetry here

$$A = 2\int_0^2 (4 - y^2)\,dy = 2\left[4y - \frac{y^3}{3} \Big|_0^2 \right]$$

$$= 2\left[8 - \frac{8}{3}\right] = 2 \cdot \frac{16}{3} = \frac{32}{3}$$

Ans.

16. Set up the integral(s) to find the area of a triangle with vertices $(-2, 5)$, $(0, -3)$ and $(5, 2)$.
A B C



Line AB

$m = \frac{5+3}{-2-0} = -4$

$y - y_1 = m(x - x_1)$

$y + 3 = -4(x - 0)$

$\boxed{y = -4x - 3}$

Line AC

$m = \frac{3}{-7}$

$y - 2 = -\frac{3}{7}(x - 5)$

$\boxed{y = -\frac{3}{7}x + \frac{15}{7} + 2}$ $\rightarrow \frac{29}{7}$

Line BC

$m = \frac{5}{5} = 1$

$y + 3 = 1(x - 0)$

$\boxed{y = x - 3}$

$$A = \int_{-2}^{0} (T - B)\,dx + \int_0^5 (T - B)\,dx$$

$$A = \int_{-2}^{0} \left(-\frac{3}{7}x + \frac{29}{7} - (-4x - 3)\right)dx + \int_0^5 \left(-\frac{3}{7}x + \frac{29}{7} - (x - 3)\right)dx$$