0. Find a gradient vector for a function $w = \frac{y\tan z}{x}$

$$\boxed{\nabla w = \langle w_x, w_y, w_z \rangle = \left\langle -\frac{y\tan z}{x^2}, \frac{\tan z}{x}, \frac{y}{x}\sec^2 z \right\rangle}$$

$\int_C f(x,y,z)\,ds$, C: $\vec{r}(t)$, $a \le t \le b$

$ds = |\vec{r}\,'(t)|\,dt$

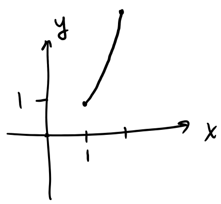
$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

$\int_C f(x,y)\,ds$, C: $y = y(x)$, $c \le x \le d$

$ds = \sqrt{1+[y'(x)]^2}\,dx$

$= \int_c^d f(x, y(x))\sqrt{1+[y'(x)]^2}\,dx$

1. Evaluate the line integral $\int_C x\,dS$, where C is the arc of the parabola $\boxed{y = x^2}$ from (1,1) to (2,4).



$ds = \sqrt{1+[y'(x)]^2}\,dx = \sqrt{1+(2x)^2}\,dx = \sqrt{1+4x^2}\,dx.$

$1 \le x \le 2.$

$\int_C x\,ds = \int_1^2 x\sqrt{1+4x^2}\,dx$

$u = 1+4x^2$
$du = 8x\,dx \Rightarrow x\,dx = \frac{du}{8}$
$x=1 \rightarrow u(1) = 1+4 = 5$
$x=2 \rightarrow u(2) = 1+4\cdot 4 = 17$

$= \frac{1}{8}\int_5^{17}\sqrt{u}\,du = \frac{1}{8}\frac{u^{3/2}}{3/2}\Big|_5^{17} = \frac{1}{\cancel{8}\,4}\cdot\frac{\cancel{2}}{3}\left(17\sqrt{17} - 5\sqrt{5}\right)$

2. Evaluate $\int_C 7y^2 z\, dS$, if C is given by $\boxed{\mathbf{r}(t) = \langle \frac{2}{3}t^3, t, t^2 \rangle}$, $0 \le t \le 1$.

$x(t) = \frac{2}{3}t^3, \quad y(t) = t, \quad z(t) = t^2 \implies x'(t) = 2t^2, \quad y'(t) = 1, \quad z'(t) = 2t$

$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$

$= \sqrt{4t^4 + 4t^2 + 1}\, dt = \sqrt{(2t^2+1)^2}\, dt = (2t^2+1)\, dt$

$\int_C 7y^2 z\, ds$ | $y = t$, $z = t^2$, $ds = (2t^2+1)\,dt$, $0 \le t \le 1$ | $= 7\int_0^1 \underbrace{(t^2)}_{y^2}\underbrace{(t^2)}_{z}\underbrace{(2t^2+1)\,dt}_{ds} = 7\int_0^1 (2t^6 + t^4)\,dt$

$= 7\left(\frac{2t^7}{7} + \frac{t^5}{5}\right)\Big|_0^1 = 7\left(\frac{2}{7} + \frac{1}{5}\right) = \ldots$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt, \quad C: \vec{r}(t),\ a\le t\le b.$$

dot product

3. Find the work done by the force field $\mathbf{F}(x,y,z) = \langle z, x, y\rangle$ in moving a particle from the point $(3,0,0)$ to the point $(0,\pi/2,3)$

   (a) along the straight line

   (b) along the helix $x = 3\cos t$, $y = t$, $z = 3\sin t$

(a) $\vec{v} = \langle -3, \frac{\pi}{2}, 3\rangle$ from $(3,0,0)$ to $(0,\frac{\pi}{2},3)$

parametric equations of the line

$$\begin{cases} x = 3-3t \\ y = \frac{\pi}{2}t \\ z = 3t \end{cases}$$

$t=0 \to (3,0,0)$

$t=1 \to (0,\frac{\pi}{2},3)$

$0\le t\le 1$

$\vec{r}(t) = \langle 3-3t, \frac{\pi}{2}t, 3t\rangle$

$\boxed{\vec{r}\,'(t) = \langle -3, \frac{\pi}{2}, 3\rangle}$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$\vec{F} = \langle z,x,y\rangle \Rightarrow \vec{F}(\vec{r}(t)) = \langle \underbrace{3t}_{z}, \underbrace{3-3t}_{x}, \underbrace{\tfrac{\pi}{2}t}_{y}\rangle$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle 3t, 3-3t, \tfrac{\pi}{2}t\rangle\cdot\langle -3, \tfrac{\pi}{2}, 3\rangle$$

$$= -9t + \frac{\pi}{2}(3-\cancel{3t}) + 3\cancel{\frac{\pi}{2}t} = \boxed{\frac{3\pi}{2} - 9t}$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1\left(\frac{3\pi}{2} - 9t\right)dt = \left(\frac{3\pi}{2}t - \frac{9t^2}{2}\right)\Big|_0^1 = \boxed{\frac{3\pi}{2} - \frac{9}{2}}$$

$x = 3\cos t$, $y = t$, $z = 3\sin t$

$\vec{r}(t) = \langle 3\cos t, t, 3\sin t\rangle$

$\vec{r}\,'(t) = \langle -3\sin t, 1, 3\cos t\rangle$

$(3,0,0) \to (0,\frac{\pi}{2},3)$

$\vec{r}(t) = \langle 3\cos t, t, 3\sin t\rangle$

$0\le t\le \frac{\pi}{2}$

$(3,0,0) = \vec{r}(0)$

$(0,\frac{\pi}{2},3) = \vec{r}(\frac{\pi}{2})$

$$\vec{F} = \langle z,x,y\rangle \longrightarrow \vec{F}(\vec{r}(t)) = \langle 3\sin t, 3\cos t, t\rangle$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle 3\sin t, 3\cos t, t\rangle\cdot\langle -3\sin t, 1, 3\cos t\rangle$$

$$= -9\sin^2 t + 3\cos t + 3t\cos t.$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2}\left(-9\sin^2 t + \underbrace{3\cos t + 3t\cos t}\right)dt$$

$$= -9\int_0^{\pi/2}\sin^2 t\,dt + 3\int_0^{\pi/2}(1+t)\cos t\,dt$$

	D	I
+	1+t	cos t
−	1	sin t

$+$	$1+t$	$\cos t$
$-$	1	$\sin t$
	0	$-\cos t$

$$= -9\int_0^{\pi/2} \frac{1-\cos 2t}{2}\,dt + 3\Big[(1+t)\sin t + \cos t\Big]_0^{\pi/2}$$

$$= -\frac{9}{2}\Big[t - \frac{1}{2}\sin 2t\Big]_0^{\pi/2} + 3\Big[\Big(1+\frac{\pi}{2}\Big)\overset{1}{\cancel{\sin\frac{\pi}{2}}} + \overset{0}{\cancel{\cos\frac{\pi}{2}}} - \overset{0}{\cancel{\sin 0}} - \overset{1}{\cancel{\cos 0}}\Big]$$

$$= -\frac{9}{2}\Big(\frac{\pi}{2} - \frac{1}{2}\overset{0}{\cancel{\sin\pi}} + \frac{1}{2}\overset{0}{\cancel{\sin 0}}\Big) + 3\Big(\cancel{1} + \frac{\pi}{2} - \cancel{1}\Big)$$

$$= \boxed{-\frac{9\pi}{4} + \frac{3\pi}{2}}$$

4. Let $\mathbf{F}(x,y) =< 3 + 2xy^2, 2x^2y >$.

(a) Show that $\mathbf{F}$ is conservative vector field.

(b) Find its potential function.

(c) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is any path from (-1,0) to (2,2).

(a) $\vec{F} = \langle P, Q\rangle$ is conservative if and only if $Q_x = P_y$

$\vec{F}(x,y) = \langle 3+2xy^2, 2x^2y\rangle$

$P(x,y) = 3+2xy^2, \quad Q(x,y) = 2x^2y$

$\frac{\partial P}{\partial y} = 4xy \xleftrightarrow[\text{match}]{} \frac{\partial Q}{\partial x} = 4xy \implies \vec{F}$ is conservative.

(b) $u(x,y)$ such that $\nabla u = \vec{F}$ or $\langle u_x, u_y\rangle = \langle 3+2xy^2, 2x^2y\rangle$

$\int u_x\,dx = \int (3+2xy^2)\,dx \longrightarrow u(x,y) = 3x + \boxed{x^2y^2} + g(y)$

$\int u_y\,dy = \int 2x^2y\,dy \rightarrow u(x,y) = \boxed{x^2y^2} + h(x)$

$$\boxed{u(x,y) = x^2y^2 + 3x}$$

(c) $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla u\cdot d\vec{r} = u(2,2) - u(-1,0) = 2^2\cdot 2^2 + 3\cdot 2 - (-1)^2 0^2 - 3(-1) = 16+6+3 = \boxed{25}$

Fundamental Theorem for Line Integrals

$\oint_C \vec{F}\cdot d\vec{r} = 0$

$$\overbrace{2x+y^2+3x^2y}^{P(x,y)},\ \overbrace{2xy+x^3+3y^2}^{Q(x,y)}$$

5. Let $\vec{F}(x,y) = < 2x + y^2 + 3x^2y, 2xy + x^3 + 3y^2 >$.

   (a) Show that $\vec{F}$ is conservative vector field.

   (b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the arc of the curve $y = x\sin x$ from $(0,0)$ to $(\pi, 0)$.

$P(x,y) = 2x+y^2+3x^2y \Longrightarrow \frac{\partial P}{\partial y} = 2y+3x^2$

$Q(x,y) = 2xy+x^3+3y^2 \longrightarrow \frac{\partial Q}{\partial x} = 2y+3x^2$

match!

$\vec{F}(x,y)$ is conservative.

Find a potential function $u(x,y)$ for $\vec{F}$.

$\nabla u = \vec{F}$ or $< u_x, u_y > = < 2x+y^2+3x^2y, 2xy+x^3+3y^2 >$

$\int u_x\,dx = \int (2x+y^2+3x^2y)\,dx \Rightarrow u(x,y) = x^2 + \boxed{y^2x + x^3y} + g(y)$

$\int u_y\,dy = \int (2xy+x^3+3y^2)\,dy \Rightarrow u(x,y) = \boxed{xy^2 + x^3y} + y^3 + h(x)$

$$\boxed{u(x,y) = xy^2 + x^3y + x^2 + y^3}$$

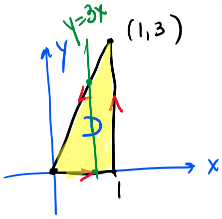
$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla u \cdot d\vec{r} = u(\pi,0) - u(0,0) = \boxed{\pi^2}$

Fundamental Theorem for line Integrals

Green's Theorem. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

6. Compute the integral $I = \oint_C \overbrace{(\cos(x^4) + xy)}^{P(x,y)}\,dx + \overbrace{(y^2e^y + x^2)}^{Q(x,y)}\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(1,3)$, and from $(1,3)$ to $(0,0)$.



positive orientation is counter clock wise.

$0 \le y \le 3x$

$0 \le x \le 1$

$$\oint_C (\cos(x^4)+xy)\,dx + (y^2e^y+x^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(y^2e^y+x^2) - \frac{\partial}{\partial y}(\cos(x^4)+xy)\right] dA$$

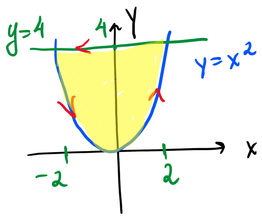
$$= \iint_D (2x - x)\,dA = \iint_D x\,dA = \int_0^1\int_0^{3x} x\,dy\,dx$$

$$= \int_0^1 xy\Big|_0^{3x} dx = \int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1$$

7. Compute the integral along the given positively oriented curve C:

$$\oint_C (y^2 - \arctan x)\,dx + (3x + \sin y)\,dy,$$

where C is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$.



$x^2 \le y \le 4$

$-2 \le x \le 2$

$$\oint_C \underbrace{(y^2 - \arctan x)}_{P}\,dx + \underbrace{(3x + \sin y)}_{Q}\,dy = \iint_D \left[Q_x - P_y\right] dA$$

$$= \iint_D \left[\frac{\partial}{\partial x}(3x + \sin y) - \frac{\partial}{\partial y}(y^2 - \arctan x)\right] dA$$

$$= \iint_D (3 - 2y)\,dA = \int_{-2}^{2}\int_{x^2}^{4} (3 - 2y)\,dy\,dx = \int_{-2}^{2} (3y - y^2)\Big|_{x^2}^{4}\,dx$$

$$= \int_{-2}^{2} (12 - 16 - 3x^2 + x^4)\,dx = \int_{-2}^{2} (-4 - 3x^2 + x^4)\,dx$$

$$= \left(-4x - \frac{\cancel{3}x^3}{\cancel{3}} + \frac{x^5}{5}\right)\Big|_{-2}^{2} = -4(2 - (-2)) - (8 - (-8)) + \frac{1}{5}(32 - (-32))$$

$$= -16 + 16 + \frac{64}{5} = \boxed{\frac{64}{5}}$$

8. Compute the integral

$$\int_C (12 - x^2y - y^3 + \tan x)\, dx + (xy^2 + x^3 - e^y)\, dy$$

where C is positively oriented boundary of the region enclosed by the circle $x^2 + y^2 = 4$. Sketch the curve C indicating the positive direction.

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$\vec{F} = \langle P, Q, R \rangle$
$\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

9. Given the vector field $\mathbf{F} = z\mathbf{i} + 2yz\mathbf{j} + (x + y^2)\mathbf{k}$.

(a) Find the divergence of the field.

(b) Find the curl of the field.

(c) Is the given field conservative? If it is, find a potential function.

(d) Compute $\int_C z\,dx + 2yz\,dy + (x + y^2)\,dz$ where C is the positively oriented curve $\overbrace{y^2 + z^2 = 4}^{\text{circle}}, x = 5$.

(e) Compute $\int_C z\,dx + 2yz\,dy + (x + y^2)\,dz$ where C consists of the three line segments: from $(0, 0, 0)$ to $(4, 0, 0)$, from $(4, 0, 0)$ to $(2, 3, 1)$, and from $(2, 3, 1)$ to $(1, 1, 1)$.

(a) $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(z)^{0} + \frac{\partial}{\partial y}(2yz)^{2z} + \frac{\partial}{\partial z}(x+y^2)^{0} = 2z$

(b) $\vec{F} = \langle P, Q, R \rangle$

$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & 2yz & x+y^2 \end{vmatrix} = \vec{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & x+y^2 \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & x+y^2 \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & 2yz \end{vmatrix}$

$= \vec{i}\left(\frac{\partial(x+y^2)}{\partial y} - \frac{\partial(2yz)}{\partial z}\right) - \vec{j}\left(\frac{\partial(x+y^2)}{\partial x} - \frac{\partial(z)}{\partial z}\right) + \vec{k}\left(\frac{\partial(2yz)}{\partial x} - \frac{\partial(z)}{\partial y}\right)$

$= \vec{i}(2y - 2y) - \vec{j}(1 - 1) + \vec{k}(0 - 0) = \vec{0} \Rightarrow \vec{F}$ is conservative.

(c) Find $u(x,y,z)$ such that $\nabla u = \vec{F}$ or $\langle u_x, u_y, u_z \rangle = \langle z, 2yz, x+y^2 \rangle$

$\int u_x\,dx = \int z\,dx \Rightarrow u(x,y,z) = zx + f(y,z)$

$\int u_y\,dy = \int 2yz\,dy \Rightarrow u = y^2z + g(x,z)$

$\int u_z\,dz = \int (x+y^2)\,dz \Rightarrow u = \boxed{xz + y^2z} + h(x,y)$

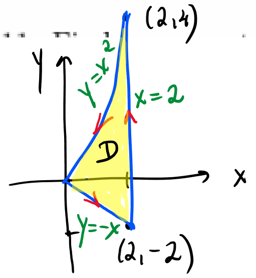
$\boxed{u(x,y,z) = xz + y^2z}$

(d) $\oint_C \vec{F}\cdot d\vec{r} = 0$

(e) $\int_C z\,dx + 2yz\,dy + (x+y^2)\,dz = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla u \cdot d\vec{r} \overset{\text{Fundamental Theorem for Line Integrals}}{=} u(1,1,1) - u(0,0,0) = 1 + 1 - 0 - 0 = 2$

$\underbrace{\boxed{(0,0,0)}}_{\text{start}} \to (4,0,0),\ (4,0,0) \to (2,3,1),\ (2,3,1) \to \underbrace{\boxed{(1,1,1)}}_{\text{end}}$

10. Given the line integral $I = \oint_C 4x^2y\,dx - (2+x)\,dy$ where C consists of the line segment from $(0,0)$ to $(2,-2)$, the line segment from $(2,-2)$ to $(2,4)$, and the part of the parabola $y = x^2$ from $(2,4)$ to $(0,0)$. Use Green's theorem to **evaluate** the given integral and **sketch** the curve C indicating the *positive direction*.



$-x \le y \le x^2$

$0 \le x \le 2$

$$\oint_C 4x^2y\,dx - (2+x)dy = \iint_D \left[\frac{-\partial}{\partial x}(2+x) - \frac{\partial}{\partial y}(4x^2y)\right] dA$$

$$= \iint_D (-1 - 4x^2)\,dA = -\int_0^2 \int_{-x}^{x^2} (1+4x^2)\,dy\,dx$$

$$= -\int_0^2 (1+4x^2)\,y\Big|_{-x}^{x^2} dx = -\int_0^2 (1+4x^2)(x^2+x)\,dx = \dots$$

11. Find a parametric representation of the following surfaces:

(a) the portion of the plane $x + 2y + 3z = 0$ inside the cylinder $x^2 + y^2 = 9$;

(b) $z + zx^2 - y = 0$;

(c) the portion of the cylinder $x^2 + z^2 = 25$ that extends between the planes $y = -1$ and $y = 3$

(a) $x+2y+3z=0$, $x^2+y^2=9$ parameter domain.

$z=-\frac{1}{3}(x+2y)$

Parametrization $\begin{cases} x=x \\ y=y \\ z=-\frac{1}{3}(x+2y) \end{cases}$,

parameter domain $D=\{(x,y) \mid x^2+y^2\le 9\}$

(b) $z+zx^2-y=0$

$y=z+zx^2$

$\begin{cases} x=x \\ y=z+zx^2 \\ z=z \end{cases}$

$z(1+x^2)-y=0$

$z=\frac{y}{1+x^2}$

$\begin{cases} x=x \\ y=y \\ z=\frac{y}{1+x^2} \end{cases}$

(c) $x^2+z^2=25$ between $y=-1$ and $y=3$

parameters t and y

$\begin{cases} x=5\cos t \\ y=y \\ z=5\sin t \end{cases}$

$\boxed{\begin{matrix} 0\le t\le 2\pi \\ -1\le y\le 3 \end{matrix}}$

S: $\vec{r}(u,v)$, $\vec{n} = \pm \vec{r}_u \times \vec{r}_v$ ← normal vector

S: $z = z(x,y)$, $\vec{n} = \pm \langle z_x, z_y, -1 \rangle$

12. Find an equation of the plane tangent to the surface $x = u, y = 2v, z = u^2 + v^2$ at the point $(1, 4, 5)$.

$\vec{r}(u,v) = \langle u, 2v, u^2+v^2 \rangle$ @ $(1,4,5)$

Find u and v such that $\langle u, 2v, u^2+v^2 \rangle = \langle 1, 4, 5 \rangle$

$u = 1$

$2v = 4 \Rightarrow v = 2.$

$\boxed{(1,4,5) = \vec{r}(1,2)}$

$\vec{r}_u = \langle 1, 0, 2u \rangle$

$\vec{r}_v = \langle 0, 2, 2v \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 2 & 2v \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & 2u \\ 2 & 2v \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 2u \\ 0 & 2v \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= -4u\vec{i} - 2v\vec{j} + 2\vec{k}$$

$\vec{n}(u,v) = \langle -4u, -2v, 2 \rangle$

$\vec{n}(1,2) = \langle -4, -4, 2 \rangle$

Tangent plane perpendicular to $\langle -4, -4, 2 \rangle$ through $(1,4,5)$

$$\boxed{-4(x-1) - 4(y-4) + 2(z-5) = 0}$$

13. Find the area of the surface with parametric equations $\boxed{x = u^2,\ y = uv,\ z = \frac{1}{2}v^2}$, $\boxed{0 \le u \le 1,\ 0 \le v \le 2.}$

S.A. $= \iint_D |\vec{n}|\, dA$, D is the parameter domain.

$\vec{n} = \pm\, \vec{r}_u \times \vec{r}_v$, $\vec{r}(u,v) = \langle u^2, uv, \frac{1}{2}v^2 \rangle$

$\vec{r}_u = \langle 2u, v, 0 \rangle$

$\vec{r}_v = \langle 0, u, v \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \vec{i}\begin{vmatrix} v & 0 \\ u & v \end{vmatrix} - \vec{j}\begin{vmatrix} 2u & 0 \\ 0 & v \end{vmatrix} + \vec{k}\begin{vmatrix} 2u & v \\ 0 & u \end{vmatrix}$$

$\vec{n} = v^2\vec{i} - 2uv\vec{j} + 2u^2\vec{k}$

$$|\vec{n}| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2} = v^2 + 2u^2$$

$$\text{S.A.} = \iint_D (v^2 + 2u^2)\, dA = \int_0^1 \int_0^2 (v^2 + 2u^2)\, dv\, du$$

$$= \int_0^1 \left(\frac{v^3}{3} + 2u^2 v \right)\Big|_0^2 du = \int_0^1 \left(\frac{8}{3} + 4u^2 \right) du$$

$$= \left(\frac{8}{3}u + \frac{4u^3}{3} \right)\Big|_0^1 = \frac{8}{3} + \frac{4}{3} = 4$$

14. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $\boxed{x^2 + y^2 = 4.}$

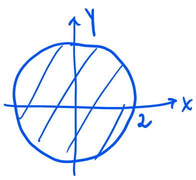
$S.A. = \iint_D |\vec{n}|\, dA$

$\vec{n} = \pm \langle 2x, 2y, -1 \rangle$

$\vec{n} = \pm \langle 2x, 2y, -1 \rangle$, $D: x^2+y^2 \le 4$

$|\vec{n}| = \sqrt{4x^2+4y^2+1}$

$S.A. = \iint_{x^2+y^2\le 4} \sqrt{4x^2+4y^2+1}\, dA$ $\left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \\ dA = r\,dr\,d\theta \end{array} \right|$ $= \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\; r\,dr\,d\theta$ $\left| \begin{array}{l} 4r^2+1 = u \\ du = 8r\,dr \\ r=2 \to u = 4(4)+1 = 17 \\ r=0 \to u = 4(0)+1 = 1 \end{array} \right|$



$= \frac{1}{8}\int_0^{2\pi}\int_1^{17} \sqrt{u}\, du\, d\theta = \frac{1}{8}\int_0^{2\pi} \frac{u^{3/2}}{3/2}\Big|_1^{17} d\theta$

$= \frac{1}{8}\cdot\frac{2}{3}\int_0^{2\pi} (17\sqrt{17} - 1)\, d\theta$

$= \frac{1}{12}(17\sqrt{17} - 1)\int_0^{2\pi} d\theta$

$= \boxed{\frac{\pi}{6}(17\sqrt{17} - 1)}$