# STAT 201 - Week-In-Review 7

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## Problem Solutions

1. Consider a discrete random variable X that assumes the integer values $0, 1, \ldots, 10$, and has probability mass function $f_X(\cdot)$. Which of the following statements concerning the probability distribution of X would be **incorrect**? Select all that apply.

   (a) $f_X(0) + f_X(1) + f_X(2) + \cdots + f_X(10) = 0.9$

   (b) $P(X \leq 3) = P(X < 3)$

   (c) $P(3 \leq X \leq 6) = f_X(3) + f_X(4) + f_X(5) + f_X(6)$

   (d) $P(3 \leq X \leq 6) = P(X \leq 6) - P(X \leq 3)$

   (e) $P(3 < X \leq 6) = f_X(4) + f_X(5) + f_X(6)$

   (f) $P(3 < X < 6) = f_X(4) + f_X(5)$

   (g) $P(X > 6) = 1 - P(X < 6)$

2. Consider the random experiment of rolling two six–faced fair dice together. Then the sample space $\mathcal{S}$ would comprise of $6 \times 6 = 36$ paired elements as

$$\mathcal{S} = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

   Let the random variable X denote the sum of the two faces obtained.

   (A) Write down the probability mass function of X.

   **Solution:** Observe that

$$X = \begin{cases} 2 & \text{if \& only if } (1,1) \text{ occurs} \\ 3 & \text{if \& only if } (1,2) \text{ or } (2,1) \text{ occurs} \\ 4 & \text{if \& only if } (1,3), (2,2) \text{ or } (3,1) \text{ occurs} \\ 5 & \text{if \& only if } (1,4), (2,3), (3,2) \text{ or } (4,1) \text{ occurs} \\ 6 & \text{if \& only if } (1,5), (2,4), (3,3), (4,2) \text{ or } (5,1) \text{ occurs} \\ 7 & \text{if \& only if } (1,6), (2,5), (3,4), (4,3), (5,2) \text{ or } (6,1) \text{ occurs} \\ 8 & \text{if \& only if } (2,6), (3,5), (4,4), (5,3) \text{ or } (6,2) \text{ occurs} \\ 9 & \text{if \& only if } (3,6), (4,5), (5,4) \text{ or } (6,3) \text{ occurs} \\ 10 & \text{if \& only if } (4,6), (5,5) \text{ or } (6,4) \text{ occurs} \\ 11 & \text{if \& only if } (5,6) \text{ or } (6,5) \text{ occurs} \\ 12 & \text{if \& only if } (6,6) \text{ occurs} \end{cases}$$

   The the probability mass function of X is given by,

| x | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_X(x) = \mathrm{P}(X = x)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 1 |

(B) What is the probability that the sum X is an even number?

**Solution:** Observe that

$$\{X \text{ is even}\} \Leftrightarrow \{X = 2\} \cup \{X = 4\} \cup \{X = 6\} \cup \{X = 8\} \cup \{X = 10\} \cup \{X = 12\}.$$

Since the above events are mutually exclusive, we obtain

$$\begin{aligned} & \mathrm{P}(X \text{ is even}) \\ = \; & \mathrm{P}(\{X = 2\} \cup \{X = 4\} \cup \{X = 6\} \cup \{X = 8\} \cup \{X = 10\} \cup \{X = 12\}) \\ = \; & \mathrm{P}(X = 2) + \mathrm{P}(X = 4) + \cdots + \mathrm{P}(X = 12) \\ = \; & \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{1}{2}. \end{aligned}$$

(C) What is the probability that the sum X is at most 4?

**Solution:** Observe that

$$\{X \leq 4\} \Leftrightarrow \{X = 2\} \cup \{X = 3\} \cup \{X = 4\}.$$

Since the above events are mutually exclusive, we obtain

$$\mathrm{P}(X \leq 4) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \mathrm{P}(X = 4) = 6/36 = 1/6.$$

(D) What is the probability that the sum X is larger than 4?

**Solution:** Since $\{X > 4\} = \{X \leq 4\}^c$, we have

$$\mathrm{P}(X > 4) = 1 - \mathrm{P}(X \leq 4) = 1 - 1/6 = 5/6.$$

3. The payoff (X) (in USD) for a game of lottery has the following probability distribution:

| Payoff (x) | Probability P(X = x) |
|---|---|
| 0 | 0.25 |
| 10 | 0.23 |
| 25 | $*$ |
| 50 | 0.15 |
| 75 | 0.15 |
| 100 | 0.05 |

Compute the expected payoff, that is, E(X), if you play the game once.

**Solution:** Observe that the missing entry in the probability mass function table of X is 0.17. Therefore,

| Payoff (x) | Probability P(X = x) | xP(X = x) |
|---|---|---|
| 0 | 0.25 | 0 |
| 10 | 0.23 | 2.3 |
| 25 | 0.17 | 4.25 |
| 50 | 0.15 | 7.5 |
| 75 | 0.15 | 11.25 |
| 100 | 0.05 | 5 |
| Total | 1 | 30.3 |

Hence, E(X) = 30.3 USD.

4. The probability distribution for the delay in hours (say, X) of the evening flight from Chicago to New York is as follows:

| x | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| P(X = x) | 0.1 | 0.1 | 0.2 | 0.3 | 0.2 | 0.1 |

(A) What is the probability that a randomly selected evening flight from Chicago to New York is delayed more than 3 hours?

(a) 0.3
(b) 0.1
(c) 0.6
(d) 0.7
(e) None of the above

(B) If we take a sample of 100 evening flights from Chicago to New York, how many of them you would expect to have a delay of at least 4 hours?

(a) 10
(b) 20
(c) 30
(d) 60
(e) It cannot be calculated

5. Decide if the following experiments are Binomial:

(a) A coin is flipped 10 times independently, and under identical conditions, and the number of heads are recorded: Yes – Binomial

(b) Cards are drawn at random, one by one and with replacement, from a well-shuffled deck of 52 cards until we get 5 jacks: No – Number of trials is not fixed

(c) 8 cards are drawn at random, one by one and with replacement, from a well-shuffled deck of 52 cards, and the number of spades are recorded: Yes – Binomial

(d) 5 individuals are drawn at random, one by one and without replacement, from a group of 22 males and 15 females to form a committee, and the number of females selected is recorded: No – Trials are dependent

(e) 95 passengers independently bought tickets for a UA-1780 flight, and the number of no shows are recorded. Assume that there is an 87.25% chance that a random customer who buys a ticket for UA-1780, actually shows up. Yes – Binomial

6. Six cards are drawn one-by-one, and with replacement from a well-shuffled deck of 52 cards.

(A) Find the probability that exactly 2 aces will appear.

**Solution:** Let the random variable X denote the number of Aces that appear. Then

$$X \sim \text{Binomial}\left(n = 6, p = \frac{1}{13}\right).$$

The p.m.f. of X is given by

$$\text{P}(X = x) = \binom{6}{x}\left(\frac{1}{13}\right)^{x}\left(\frac{12}{13}\right)^{6-x}, \text{ for } x = 0, 1, \dots, 6.$$

The probability that exactly 2 aces will appear is:

$$\text{P}(X = 2) = \binom{6}{2}\left(\frac{1}{13}\right)^{2}\left(\frac{12}{13}\right)^{6-2} = \binom{6}{2}\left(\frac{1}{13}\right)^{2}\left(\frac{12}{13}\right)^{4}.$$

(B) Find the The probability that at most 2 aces will appear.

**Solution:** The probability that at most 2 aces will appear is:

$$\begin{aligned} \text{P}(X \leq 2) &= \text{P}(X = 0) + \text{P}(X = 1) + \text{P}(X = 2) \\ &= \binom{6}{0}\left(\frac{1}{13}\right)^{0}\left(\frac{12}{13}\right)^{6-0} + \binom{6}{1}\left(\frac{1}{13}\right)^{1}\left(\frac{12}{13}\right)^{6-1} + \binom{6}{2}\left(\frac{1}{13}\right)^{2}\left(\frac{12}{13}\right)^{6-2} \\ &= \binom{6}{0}\left(\frac{1}{13}\right)^{0}\left(\frac{12}{13}\right)^{6} + \binom{6}{1}\left(\frac{1}{13}\right)^{1}\left(\frac{12}{13}\right)^{5} + \binom{6}{2}\left(\frac{1}{13}\right)^{2}\left(\frac{12}{13}\right)^{4}. \end{aligned}$$

(C) Find the expected number of aces to be drawn in those 6 draws.

**Solution:** The expected number of aces to be drawn in those 6 draws is:

$$\text{E}(X) = np = \frac{6}{13} \approx 0.46154.$$

7. Air-USA has a policy of booking as many as 25 persons on a small airplane that can seat only 22 passengers. Past studies have revealed that only 85% of the booked passengers actually arrive for the flight. Find the probability that if Air-USA books 25 seats, not enough seats will be available. Assume that the seats were booked independently of each other.

**Solution:** Let the r.v. X denote the number of passengers who actually arrive for the flight when 25 passenger seats were booked. Then,

$$X \sim \text{Binomial}(n = 25, p = 0.85).$$

Therefore, the p.m.f. of X is given by

$$\mathrm{P}(X = x) = \binom{25}{x}(0.85)^{x}(0.15)^{25-x}, \quad \text{for } x = 0, 1, \ldots, 25.$$

Note that the seating capacity of the flight is for 22 passengers, and 25 seats were booked. Hence, the event that not enough seats will be available can occur if and only if X exceeds 22, that is, if and only if $X \geq 23$.

Therefore, the required probability:

$$\begin{aligned}
\mathrm{P}(X \geq 23) &= \mathrm{P}(X = 23) + \mathrm{P}(X = 24) + \mathrm{P}(X = 25) \\
&= \binom{25}{23}(0.85)^{23}(0.15)^{25-23} + \binom{25}{24}(0.85)^{24}(0.15)^{25-24} + \binom{25}{25}(0.85)^{25}(0.15)^{25-25} \\
&= \binom{25}{23}(0.85)^{23}(0.15)^{2} + \binom{25}{24}(0.85)^{24}(0.15)^{1} + \binom{25}{25}(0.85)^{25}(0.15)^{0}.
\end{aligned}$$

8. Suppose X is a continuous random variable with probability density function (PDF):

$$f(x) = \begin{cases} c & \text{if } 0 \leq x \leq 0.5, \\ 0 & \text{otherwise,} \end{cases}$$

where $c > 0$ is an unknown, but fixed real number.

Select the CORRECT statement from the following:

(a) $\mathrm{P}(X = 0.23) = c$

(b) $0 \leq f(x) \leq 1$ because it is a probability.

(c) The constant c must be 2 so that the function f is a valid PDF.

(d) $\mathrm{P}(0.17 \leq X < 0.32) = f(0.32) - f(0.17)$

(e) $\mathrm{P}(x > 0.32) = 1 - f(0.32)$

9. For 2012, the SAT math test had an approximate normal distribution with mean 514 and standard deviation 117. The ACT math test is an alternate to the SAT and is approximately normally distributed with mean 21 and standard deviation of 5.3.

If a person took the SAT math test and scored 700 and a second person took the ACT math test and scored 30, who did better with respect to the test they took?

**Solution:** The $z-$score corresponding to the SAT score of 700:

$$z_{\mathrm{SAT}} = \frac{700 - 514}{117} = 1.58974$$

The $z-$score corresponding to the ACT score of 30:

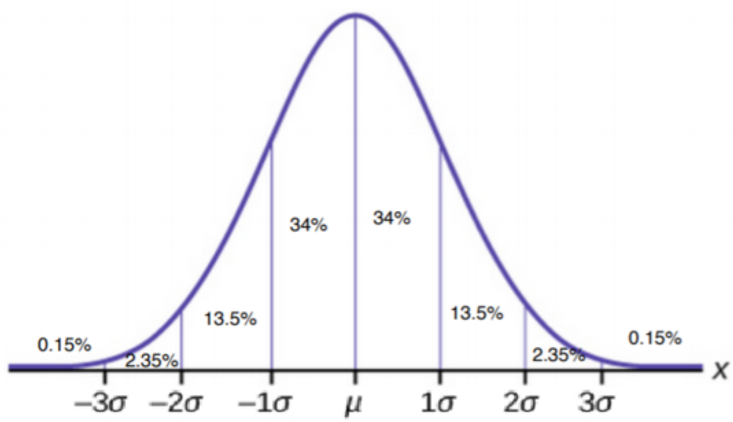
$$z_{\mathrm{ACT}} = \frac{30 - 21}{5.3} = 1.69811$$

Since $z_{\mathrm{ACT}} > z_{\mathrm{SAT}}$, the person scoring 30 in the ACT test did better as compared to the person scoring 700 in the SAT test.

10. The heights (X) of adult women in inches approximately follows a normal distribution with mean $\mu = 65$ and standard deviation $\sigma = 4$.

    What percentage of women in the population have heights between 61 and 73 inches?

    **Solution:**




$$\mathrm{P}(61 \leq X \leq 73) = \mathrm{P}(-1 \leq Z \leq 2) = 0.34 + 0.34 + 0.135 = 0.815$$

    Hence, the required percentage is 81.5%.

11. It is known that the resistance of carbon resistors is approximately normally distributed with $\mu = 1200$ ohms and $\sigma = 120$ ohms. If one resistor is randomly selected from a shipment, what is the probability that the its resistance will be less than 1250?

    **Solution:** Let the random variable X denote the resistance of the randomly chosen resistor. Then, according to the problem,

$$X \overset{a}{\sim} \mathrm{N}(\mu = 1200,\ \sigma = 120).$$

    We need to find

$$\mathrm{P}(X < 1250) = \mathrm{P}\left(Z < \frac{1250 - \mu}{\sigma}\right) = \mathrm{P}\big(Z < 0.41667\big) \approx \mathrm{P}\big(Z < 0.42\big) = .66276.$$

12. An expert witness for a paternity lawsuit testifies that the length of a pregnancy is approximately normally distributed with a mean of 280 days and a standard deviation of 13 days.

    An alleged father was out of the country from 240 to 306 days before the birth of the child, so the pregnancy would have been less than 240 days or more than 306 days long if he was the father. The birth was uncomplicated, and the child needed no medical intervention.

    What is the probability (up to four decimal places) that he was NOT the father?

    **Solution:** Let the random variable X denote the length of pregnancy for a randomly selected woman. According to the problem,

$$X \overset{a}{\sim} \mathrm{N}(\mu = 280,\ \sigma = 13).$$

Using the z-table, we obtain

$$\begin{aligned}
\mathrm{P}(240 \le X \le 306) &= \mathrm{P}\left(\frac{240-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{360-\mu}{\sigma}\right) \\
&= \mathrm{P}\big(-3.0769 \le Z \le 2\big) \\
&= \mathrm{P}\big(Z \le 2\big) - \mathrm{P}\big(Z \le -3.0769\big) \\
&\approx \mathrm{P}\big(Z \le 2\big) - \mathrm{P}\big(Z \le -3.08\big) \\
&= 0.97725 - 0.00104 \\
&= 0.97621.
\end{aligned}$$

Thus, the required probability that the alleged person was not the father of the child is approximately 0.97621.

13. Suppose, the systolic blood pressure (given in millimeters) of males follows approximately a normal distribution with mean $\mu = 125$ and standard deviation $\sigma = 14$. Find the probability that the systolic blood pressure of a randomly selected male will lie in between 94 and 139, both inclusive.

    **Solution:** Let the random variable X denote the systolic blood pressure of a randomly selected male. Then, according to the problem,

$$X \sim \mathrm{N}(\mu = 125,\ \sigma = 14).$$

Using the z-table, we obtain

$$\begin{aligned}
\mathrm{P}(94 \le X \le 139) &= \mathrm{P}\left(\frac{94-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{139-\mu}{\sigma}\right) \\
&= \mathrm{P}\big(-2.2143 \le Z \le 1\big) \\
&= \mathrm{P}\big(Z \le 1\big) - \mathrm{P}\big(Z \le -2.2143\big) \\
&\approx \mathrm{P}\big(Z \le 1\big) - \mathrm{P}\big(Z \le -2.21\big) \\
&= 0.84134 - 0.01355 \\
&= 0.82779.
\end{aligned}$$

14. The Cowboys and the Giants are playing against each other next weekend. Assume the time it takes until the first touchdown is scored in an NFL game is normally distributed with a mean $\mu = 8.5$ minutes and standard deviation $\sigma = 0.75$ minutes. Suppose the time the first touchdown is scored in the game is in the top 5 percent. What is the time associated with being in the top 5 percent?

    **Solution:** Let the random variable X denote the time until the first touchdown is scored in an NFL game. Then, $X \sim \mathrm{N}(\mu = 8.5,\ \sigma = 0.75)$.

    We need to find the time such that the area under the curve towards its right is 5%, that is, we need to compute the 95$-$th percentile of X.

Here p = 0.95, so that the 95−th percentile of X is given by

$$\begin{aligned} x_{0.05} &= \mu + \sigma \cdot z_{0.05} \\ &= 8.5 + 0.75 \times 1.64485 \quad [\text{since } z_{0.05} = 1.64485] \\ &= 9.7336375 \\ &\approx 9.734. \end{aligned}$$

Hence, the required first touchdown time is 9.734 minutes.