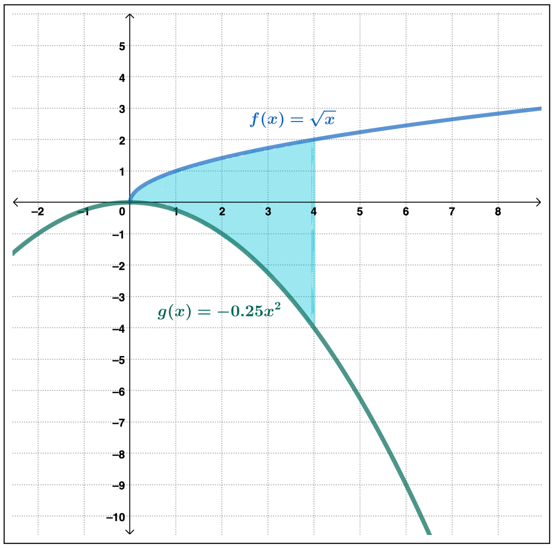
Session ~~1~~ 11: Review for Final Exam

Section 4.6

1. Find the area between the two curves $f(x) = \sqrt{x}$ and $g(x) = -0.25x^2$ on the interval $[0, 4]$.



$$\int_0^4 \sqrt{x} - (-.25x^2)\, dx$$

$$= \int_0^4 x^{1/2} + .25x^2\, dx = \frac{2}{3}x^{3/2} + \frac{1}{12}x^3 \Big]_0^4$$

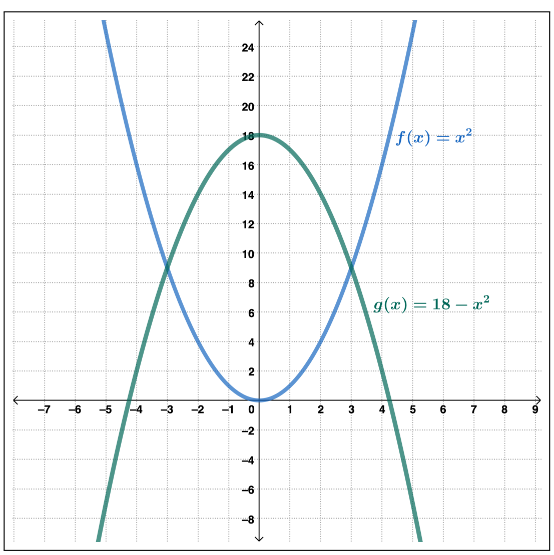
$$= \frac{2}{3}(4)^{3/2} + \frac{1}{12}(4)^3 - 0$$

$$= \frac{2}{3}(8) + \frac{64}{12}$$

$$= \frac{16}{3} + \frac{16}{3}$$

$$= \boxed{\frac{32}{3}}$$

2. Find the area between the two curves $f(x) = x^2$ and $g(x) = 18 - x^2$.



$x^2 = 18 - x^2$

$2x^2 = 18$

$x^2 = 9$

$x = \pm 3$

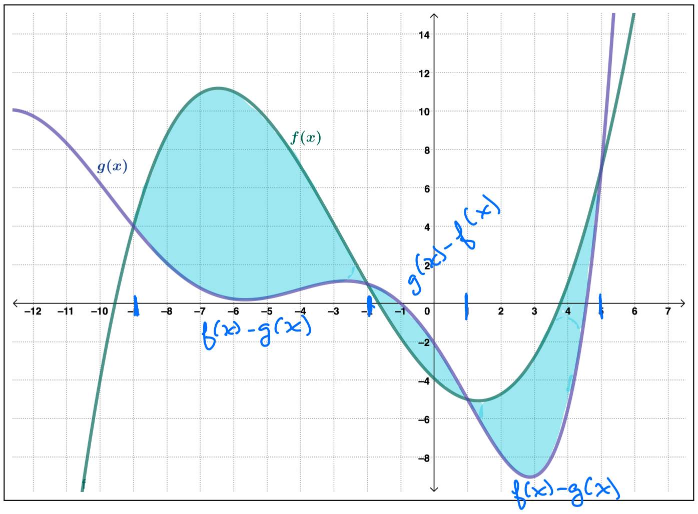
$$\int_{-3}^{3} \underbrace{g(x)}_{\text{top}} - \underbrace{f(x)}_{\text{bottom}}\, dx$$

$$= \int_{-3}^{3} 18 - x^2 - x^2\, dx = \int_{-3}^{3} 18 - 2x^2\, dx$$

$$\int_{-3}^{3}(18-2X^2)dX$$

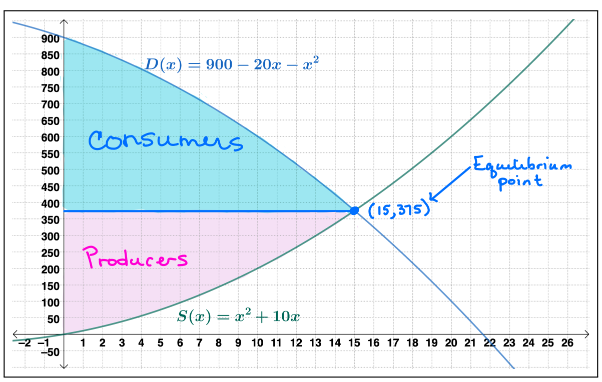
72

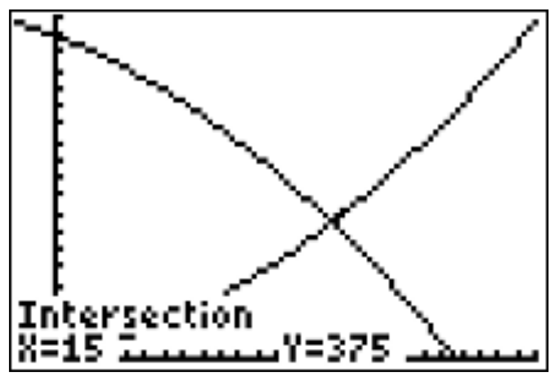
3. Given the graphs of $f(x)$ and $g(x)$ below, find the integral that represents the area between the two curves. in terms of f(x) and g(x)



$$\int_{-9}^{-2} (f(x) - g(x))\,dx + \int_{-2}^{1} (g(x) - f(x))\,dx + \int_{1}^{5} (f(x) - g(x))\,dx$$

4. Suppose the supply function of a certain item is given by $S(x) = x^2 + 10x$ (in dollars) and the demand function is given by $D(x) = 900 - 20x - x^2$ (in dollars) where x is the number of items produced and sold.



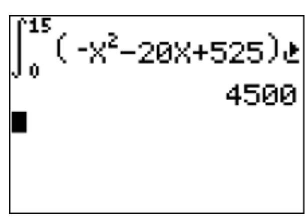


(a) Find the consumers' surplus.

$$\int_0^{15} (900 - 20x - x^2 - 375)\,dx = \int_0^{15} (-x^2 - 20x + 525)\,dx$$

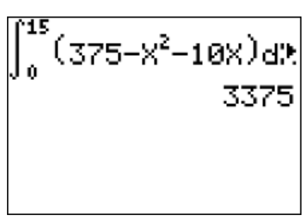
use calculator

$= \$4500$



(b) Find the producers' surplus.

$$\int_0^{15} 375 - x^2 - 10x \,dx = \$3375$$



Concepts from Exam 1

5. Given $h(x)$ below, answer the questions that follow.

$$h(x) = \begin{cases} 2e^{x+5} & x < -5 \\ \dfrac{-20}{x-5} & -5 \le x < 4 \\ \dfrac{(3x-2)(x-8)}{x-8} & 4 \le x \le 10 \end{cases}$$

(a) $\lim_{x \to -5^+} h(x)$. $= \lim_{x \to -5^+} \dfrac{-20}{x-5} = \dfrac{-20}{-5-5} = \dfrac{-20}{-10} = 2$

(b) $\lim_{x \to -5^-} h(x)$ $= \lim_{x \to -5^-} 2e^{x+5} = 2e^{-5+5} = 2e^{0} = 2(1) = 2$

(c) $\lim_{x \to 4} h(x)$ DNE

$\lim_{x \to 4^-} h(x) = \lim_{x \to 4^-} \dfrac{-20}{x-5} = \dfrac{-20}{4-5} = \dfrac{-20}{-1} = 20$

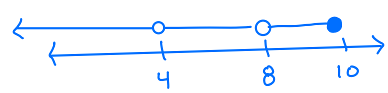
$\lim_{x \to 4^+} h(x) = \lim_{x \to 4^+} (3x-2) = 3(4)-2 = 12-2 = 10$

(d) the intervals on which $h(x)$ is continuous.

$(-\infty, 4) \cup (4, 8) \cup (8, 10]$

$$h(x) = \begin{cases} \checkmark\, \underbrace{2e^{x+5}}_{\text{continuous everywhere}} & x < -5 \,\checkmark \\ \checkmark\, \underbrace{\dfrac{-20}{x-5}}_{x \ne 5 \text{ but } x=5 \text{ is not on } -5 \le x < 4} & -5 \le x < 4 \\ \dfrac{(3x-2)\cancel{(x-8)}}{\cancel{x-8}} & 4 \le x \le 10 \end{cases}$$

$x = 8$ is hole and $x = 8$ is on $4 \le x \le 10$.



Check $x = 4$

$\lim_{x \to 4^-} \dfrac{-20}{x-5} = \dfrac{-20}{4-5} = 20$

$\lim_{x \to 4^+} (3x-2) = 3(4)-2 = 12-2 = 10$

$20 \ne 10$ so $\lim_{x \to 4} h(x)$ DNE, not continuous at $x = 4$

Check $x = -5$

- from (a) and (b) above we see $\lim_{x \to -5^-} h(x) = \lim_{x \to -5^+} h(x) = 2$. So $\lim_{x \to -5} h(x) = 2$
- $h(-5) = 2$
- $h(x)$ is continuous at $x = -5$.

6. Given the information about $p(x)$ below, find the equation of the line tangent to $p(x)$ at $x = 2$.

- $p'(x) = \sqrt{2x+5}$ $\longleftarrow$ Slope of tangent $p'(2) = \sqrt{2 \cdot 2 + 5}$ $= 3 \leftarrow m$
- $p''(x) = (2x+5)^{-1/2}$
- $p(-2.5) = 3$
- $p(2) = 12.$ $\longleftarrow$ point of tangency (x_1, y_1)

$$y - 12 = 3(x-2)$$

$$y = 3x - 6 + 12 = 3x + 6$$

7. Find the limits specified below. If the limit does not exist but has infinite behavior, use limit notation to describe the infinite behavior.

(a) $\lim\limits_{x\to 3} \dfrac{2x^2 - 18}{4(x-3)(x-2)} = \lim\limits_{x\to 3} \dfrac{2(x^2-9)}{4(x-3)(x-2)} = \lim\limits_{x\to 3} \dfrac{2(x+3)\cancel{(x-3)}}{4\cancel{(x-3)}(x-2)}$

$$= \lim_{x\to 3} \frac{2(x+3)}{4(x-2)}$$

$$= \frac{2(3+3)}{4(3-2)}$$

$$= \frac{2(6)}{4} = \frac{12}{4} = 3$$

(b) $\lim\limits_{x\to\infty} \dfrac{x^4 + x^2 + 1}{3x^3 + x - 3} = \lim\limits_{x\to\infty} \dfrac{\frac{x^4}{x^3} + \frac{x^2}{x^3} + \frac{1}{x^3}}{\frac{3x^3}{x^3} + \frac{x}{x^3} - \frac{3}{x^3}}$

$$= \lim_{x\to\infty} \frac{x + \cancel{\frac{1}{x}}^{\,0} + \cancel{\frac{1}{x^3}}^{\,0}}{3 + \cancel{\frac{1}{x^2}}^{\,0} - \cancel{\frac{3}{x^3}}^{\,0}} \to \infty$$

8. Find the horizontal asymptotes for $f(x) = \dfrac{4 + e^x - 5e^{-x}}{3e^{-x} + 8}$.

$$\lim_{x \to \infty} \frac{4 + \cancel{e^{x}}^{\,\infty} - 5\cancel{e^{-x}}^{\,0}}{3\cancel{e^{-x}}^{\,\to 0} + 8} \longrightarrow \infty$$

, no horizontal asy. as $x \to \infty$

$$\lim_{x \to -\infty} \frac{\frac{4}{e^{-x}} + \frac{e^{x}}{e^{-x}} - \frac{5e^{-x}}{e^{-x}}}{\frac{3e^{-x}}{e^{-x}} + \frac{8}{e^{-x}}} = \lim_{x \to -\infty} \frac{4\cancel{e^{x}}^{\,0} + \cancel{e^{2x}}^{\,0} - 5}{3 + 8\cancel{e^{x}}^{\,\to 0}} = -\frac{5}{3}$$

$$\boxed{\text{horizontal asymptote at } y = -\frac{5}{3}}$$

9. Use the limit definition of the derivative to find $f'(x)$ given $f(x) = x^2 + 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}}$$

$$= 2x$$

$$f(x+h) = (x+h)^2 + 1$$

$$= x^2 + 2xh + h^2 + 1$$

$$f(x) = x^2 + 1$$

$$f(x+h) - f(x) =$$

$$\cancel{x^2} + 2xh + h^2 \cancel{+1} \cancel{-x^2} \cancel{-1}$$

$$2xh + h^2$$

Concepts from Exam 2

10. Find the derivatives of the following functions. You do not need to simplify your answers. Apply the correct derivative rules and leave in unsimplified form.

(a) $q(x) = \log_4 \left(x^2 + e^{3x} \right)$

$$q'(x) = \frac{2x + 3e^{3x}}{(x^2 + e^{3x}) \ln 4}$$

(b) $f(x) = 4^{2x^3+4x-1}$

$$f'(x) = 4^{2x^3+4x-1} (\ln 4)(6x^2 + 4)$$

(c) $r(x) = \dfrac{(x^2+3)^4}{e^{3x^2}}$

$$r'(x) = \frac{e^{3x^2}\left(4(x^2+3)^3(2x)\right) - (x^2+3)^4\left(e^{3x^2} \cdot 6x\right)}{\left(e^{3x^2}\right)^2}$$

(d) $t(x) = 2^{5x^3} \left(\ln(2x^2 + 4x - 1) \right)$

$$t'(x) = 2^{5x^3} \left(\frac{4x+4}{2x^2+4x-1} \right) + \ln(2x^2+4x-1) \cdot 2^{5x^3} (\ln 2)(15x^2)$$

11. Given $16x^2 + \frac{1}{2}y^2 - 3xy = 16$, find $\frac{dy}{dx}$ and then use that to find the equation of the line tangent to the given curve at the point $(1, 6)$.

$$32x + y\frac{dy}{dx} - \left(3x\frac{dy}{dx} + 3y\right) = 0$$

$$32x + y\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$$

$$y\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 32x$$

$$\frac{dy}{dx}\left(y - 3x\right) = 3y - 32x$$

$$\frac{dy}{dx} = \frac{3y - 32x}{y - 3x}$$

when $x = 1$ and $y = 6$, $\frac{dy}{dx} = \frac{3(6) - 32(1)}{6 - 3(1)}$

$$= \frac{18 - 32}{6 - 3}$$

$$= \frac{-14}{3}$$

$$\boxed{y - 6 = -\frac{14}{3}(x - 1)}$$

12. Given $h(x) = \dfrac{(f(x))^2}{3e^x}$, $f(0) = 5$ and $f'(0) = -1$ find $h'(0)$.

$$h'(x) = \frac{3e^x\left(2f(x)\,f'(x)\right) - (f(x))^2 \cdot 3e^x}{(3e^x)^2}$$

$$h'(0) = \frac{3e^0\left(2f(0)\,f'(0)\right) - (f(0))^2 \cdot 3e^0}{(3e^0)^2}$$

$$= \frac{3(2)(5)(-1) - (5)^2(3)}{3^2}$$

$$= \frac{-30-75}{9}$$

$$= \frac{-105}{9}$$

$$= \boxed{\frac{-35}{3}}$$

13. The cost equation (in dollars) for a certain company to produce x items is given by $C(x) = \sqrt{x}(x+12)+50$ for $0 \le x \le 500$. Find the ***approximate*** cost of producing the 36th item.

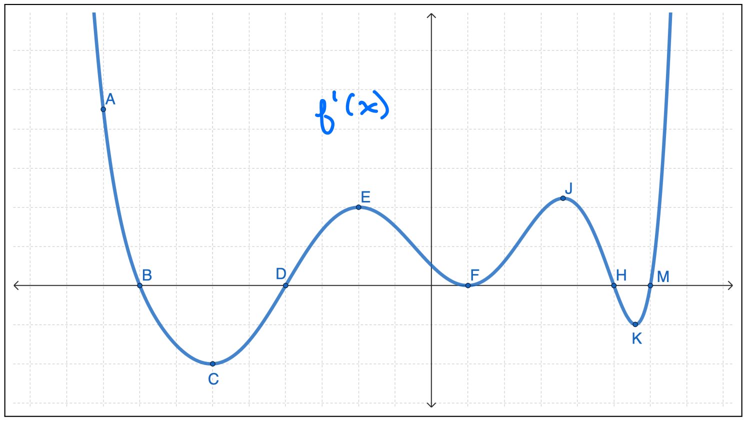
find $c'(35)$

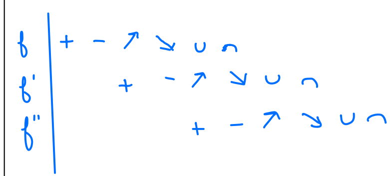
use nDeriv on calculator

$c'(35) = 9.888304$

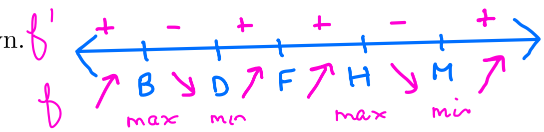
$\boxed{\$\ 9.89}$

14. A graph of $f'(x)$ is given below. Use the graph of $f'(x)$ to answer questions that follow.



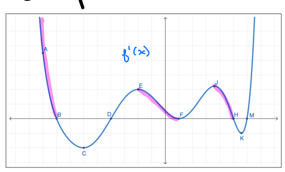


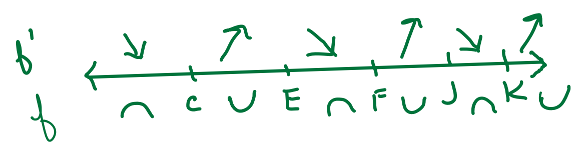
(a) Determine the intervals where $f(x)$ is increasing and concave down.



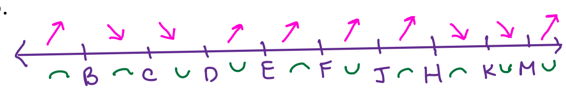
$(-\infty, B)$, (E, F), (J, H)

($f'(x)$ must be positive and decreasing.



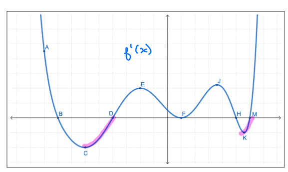


(b) Determine the intervals where $f(x)$ is decreasing and concave up.



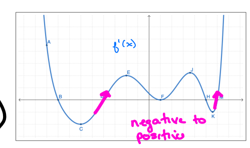
(C, D), (K, M)

($f'(x)$ must be negative and increasing)

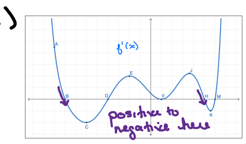


(c) Determine the x-value(s) at which $f(x)$ has local extrema.

local min : $x = D, M$ ($f'(x)$ changes from negative to positive)



local max: $x = B, H$ ($f'(x)$ changes from positive to negative)



(d) Determine the x-values at which $f(x)$ has points of inflection.

$x = C, E, F, J, K$ (local max + min of $f'(x)$)

15. Given $g(x)$ is continuous on its domain of $(-\infty,4)\cup(4,\infty)$, $g'(x)=\dfrac{2(x-2)(x+5)}{(x-4)^2}$ and $g''(x)=\dfrac{-22x+16}{(x-4)^3}$:

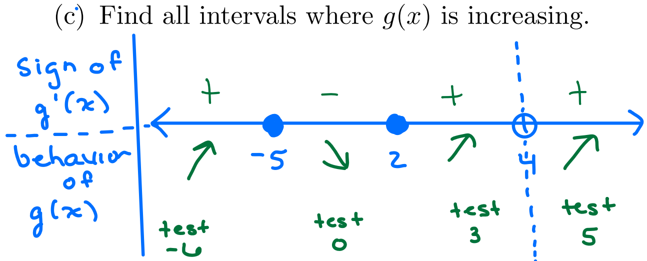
(a) Find all partition values for $g'(x)$.

$\boxed{x = 2, -5, 4}$

(b) Find all critical values for $g(x)$.

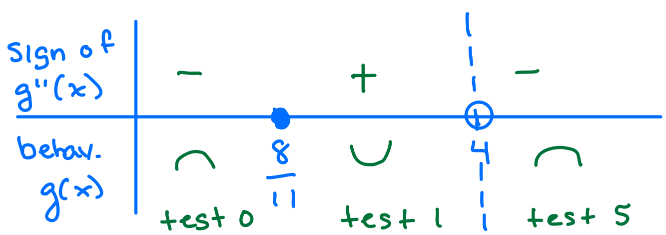
$\boxed{x = 2, -5}$

(c) Find all intervals where $g(x)$ is increasing.



$\boxed{(-\infty,-5),\ (2,4),\ \text{and } (4,\infty)}$

(d) Find all intervals where $g(x)$ is concave down.



$-22x+16=0$

$x = \frac{-16}{-22} = \frac{8}{11}$

$\boxed{(-\infty, 8/11) \text{ and } (4,\infty)}$

(e) Find all x-values where $g(x)$ has point(s) of inflection.

$\boxed{x = 8/11}$

(f) Find all x-values where $g(x)$ has local extrema. Be sure to specify the type of extrema in your answer.

local max @ $x = -5$

local min @ $x = 2$

Concepts from Exam 3

16. Find $\int \frac{t^2+1}{\sqrt{t}}\,dt$ $= \int \frac{t^2}{t^{1/2}} + \frac{1}{t^{1/2}}\,dt$

$$= \int t^{3/2} + t^{-1/2}\,dt$$

$$\boxed{= \frac{2}{5}t^{5/2} + 2t^{1/2} + C}$$

17. Evaluate $\int 3xe^{x^2+5}\,dx.$ $= \int \frac{3}{2}e^{u}\,du$

$u = x^2+5$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\frac{3}{2}du = 3x\,dx$

$$= \frac{3}{2}e^{u} + C$$

$$\boxed{= \frac{3}{2}e^{x^2+5} + C}$$

18. Find $\int_1^b \left(\frac{2}{x^2}+6\right)\,dx$, where b is a real number and $b>1$.

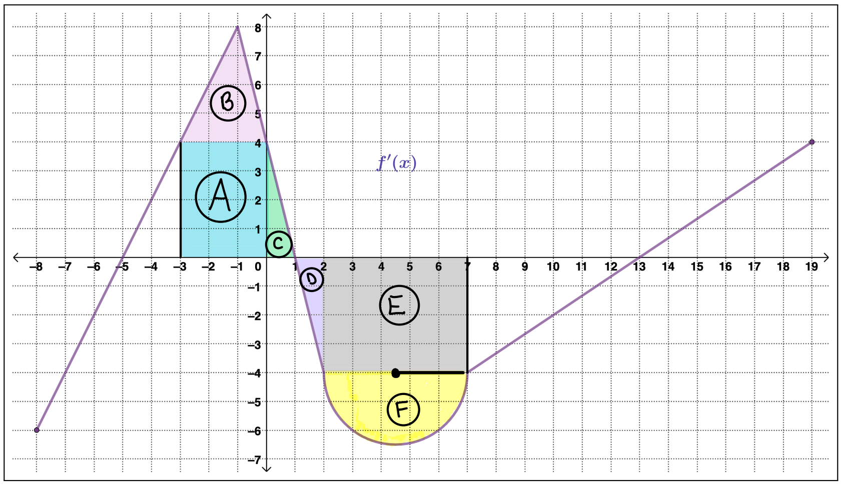
$$\int_1^b (2x^{-2}+6)\,dx = -2x^{-1} + 6x\Big]_1^b$$

$$= -2(b)^{-1} + 6(b) - \left(-2(-1)^{-1} + 6(1)\right)$$

$$= -\frac{2}{b} + 6b - (2+6)$$

$$= \boxed{-\frac{2}{b} + 6b - 8}$$

19. Use the graph of $f'(x)$ below to find $\int_{-3}^{7} f'(x)\,dx$.



Areas of Regions

A: $(3)(4) = 12$

B: $\frac{1}{2}(3)(4) = 6$

C: $\frac{1}{2}(1)(4) = 2$

D: $\frac{1}{2}(1)(4) = 2$

E: $(5)(4) = 20$

F: $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2.5)^2 = \frac{1}{2}\pi(6.25) = 3.125\pi$

$$\int_{-3}^{7} f'(x)\,dx = \underbrace{\text{area region A} + \text{area region B} + \text{area region C}}_{\text{above x-axis}} \underbrace{- \text{area region D} - \text{area region E} - \text{area region F}}_{\text{below x-axis}}$$

$$= 12 + 6 + 2 - 2 - 20 - 3.125\pi$$

$$= \boxed{-2 - 3.125\pi}$$

20. Find all abosolute extrema of $g(x) = 2x^4 - 4x^2 + 1$ on $[-2, 3]$.

$g'(x) = 8x^3 - 8x = 8x(x^2-1)$

$= 8x(x+1)(x-1)$

$x = 0, -1, 1$

x	$g(x)$
-2	17
-1	-1
0	1
1	-1
3	127 ← max

min

absolute max of 127 at $x=3$

absolute min of -1 at $x = 1, -1$.

21. A small business sells wind chimes and has a revenue function (in dollars) of

$$R(x) = -\frac{1}{30}x^3 + 7x^2,$$

where x is the number of wind chimes produced and sold, and a cost function (in dollars) of

$$C(x) = 130x + 3000.$$

Use techniques of calculus to find the number of wind chimes the company would have to produce to **maximize profit** knowing the company will produce between 0 and 170 wind chimes.

$$P(x) = R(x) - C(x)$$

$$= -\frac{1}{30}x^3 + 7x^2 - 130x - 3000$$

$$P'(x) = -\frac{3}{30}x^2 + 14x - 130$$

$$0 = -\frac{1}{10}x^2 + 14x - 130$$

$$0 = -\frac{1}{10}(x^2 - 140x + 1300)$$

$$0 = -\frac{1}{10}(x-130)(x-10)$$

$$x = 130, 10$$

	x	$P(x)$
	0	-3000
	10	-3633.33
max	130	25166.67
	170	13433.33

The company should produce 130 windchimes.