Math 151
Week-In-Review 4
Exam 1 Review
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Problem Statements

1. Let $\boldsymbol{a} = \langle -3, 7\rangle$, $\boldsymbol{b} = 4\boldsymbol{i} - \boldsymbol{j}$, and $\boldsymbol{c} = 2\boldsymbol{a} - 3\boldsymbol{b}$, find the following.

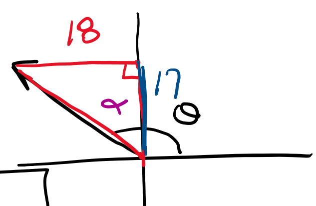
$\vec{b} = \langle 4, -1 \rangle$

(a) $\boldsymbol{c}$

$$= 2\langle -3,7\rangle - 3\langle 4,-1\rangle$$
$$= \langle -6, 14\rangle - \langle 12, -3\rangle$$
$$= \langle -6-12,\ 14-(-3)\rangle = \boxed{\langle -18, 17\rangle}$$

(b) $|\boldsymbol{c}|$

$$|\vec{c}| = \sqrt{(-18)^2 + (17)^2}$$



(c) The angle c makes with the positive $x-$axis.

$$\tan(\alpha) = \frac{18}{17}$$
$$\alpha = \arctan\left(\frac{18}{17}\right)$$

$$\boxed{\begin{aligned}\theta &= \alpha + \frac{\pi}{2}\\ &= \alpha^{\circ} + 90^{\circ}\end{aligned}}$$

(d) $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{-3(4) + 7(-1)}{\sqrt{(-3)^2 + 7^2}} = \frac{-12-7}{\sqrt{9+49}} = \boxed{\frac{-19}{\sqrt{58}}}$$

(e) $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$

Unit Vector in direction of $\vec{a}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|} = \frac{-19}{\sqrt{58}}\,\frac{\langle -3,7\rangle}{\sqrt{58}} = \boxed{\frac{-19}{58}\langle -3,7\rangle} =$$

(f) The angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos(\theta)$$
$$-19 = \sqrt{58}\,\sqrt{4^2+(-1)^2}\,\cos\theta$$
$$-19 = \sqrt{58}\,\sqrt{17}\,\cos\theta$$

$$\cos(\theta) = \frac{-19}{\sqrt{58}\,\sqrt{17}}$$

$$\boxed{\theta = \arccos\left(\frac{-19}{\sqrt{58}\,\sqrt{17}}\right)}$$

2. Determine the value(s) of x for which the vectors $\langle x, -2\rangle$ and $\langle x-9, 5\rangle$ are perpendicular.

$\langle x, -2\rangle \cdot \langle x-9, 5\rangle$

$x(x-9) + (-2)(5) = 0$

$x^2 - 9x - 10 = 0$

$(x-10)(x+1) = 0$

$x - 10 = 0 \qquad x+1 = 0$

$\boxed{x = 10 \qquad x = -1}$

3. Determine the work done by a Force $\boldsymbol{F} = \langle 12, 7\rangle$ as it moves an object from the point $(-1, 2)$ to the point $(8, 6)$.

$\vec{d} = \langle 8-(-1), 6-2\rangle$

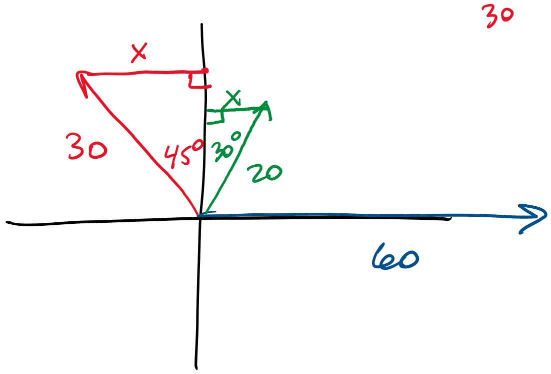
$\vec{d} = \langle 9, 4\rangle$

$W = \vec{F} \cdot \vec{d} = |\vec{F}|\,|\vec{d}| \cos(\theta)$

$= 12(9) + 7(4) = 108 + 28 = \boxed{136}$

4. A cruise ship is headed N 45° W at 30 mph. Daredevil Darren is driving a motorcycle on deck due East at 60 mph. Just as Darren is about to jump off of the ship, a wind blows in, pushing 20 mph at N 30° E. What is Darren's speed as he jumps off the ship? Find the direction he is headed as he leaves the ship, as a bearing.

$$\frac{x}{30} = \sin(45)$$



$$\vec{S} = \langle -30\sin(45),\ 30\cos(45)\rangle$$

$$\vec{W} = \langle 20\sin(30),\ 20\cos(30)\rangle$$

$$\vec{D} = \langle 60,\ 0\rangle$$

$$\vec{S} = \left\langle -30\left(\frac{\sqrt{2}}{2}\right),\ 30\left(\frac{\sqrt{2}}{2}\right)\right\rangle = \langle -15\sqrt{2},\ 15\sqrt{2}\rangle$$

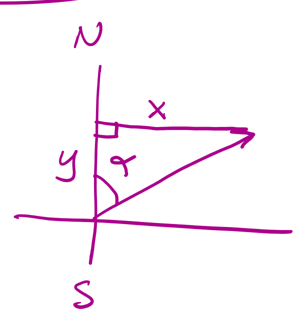
$$\vec{W} = \left\langle 20\left(\frac{1}{2}\right),\ 20\left(\frac{\sqrt{3}}{2}\right)\right\rangle = \langle 10,\ 10\sqrt{3}\rangle$$

$$\vec{V} = \langle 60 - 15\sqrt{2} + 10,\ 0 + 15\sqrt{2} + 10\sqrt{3}\rangle$$

$$= \langle 70 - 15\sqrt{2},\ 15\sqrt{2} + 10\sqrt{3}\rangle$$

$$\boxed{\text{Speed: } |\vec{V}| = \sqrt{(70-15\sqrt{2})^2 + (15\sqrt{2}+10\sqrt{3})^2}}$$

$$\tan(\alpha) = \frac{x}{y}$$



$$\boxed{\alpha = \arctan\left(\frac{70-15\sqrt{2}}{15\sqrt{2}+10\sqrt{3}}\right)}$$

Direction: N α° E

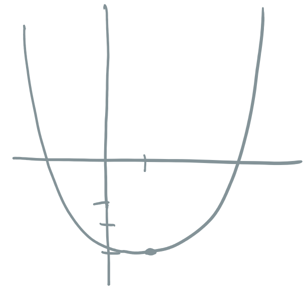
5. Find a Cartesian Equation of the curve $x = 4t^2 - 3$ and $y = 2t + 1$. Sketch the curve and indicate the direction of motion as t increases.

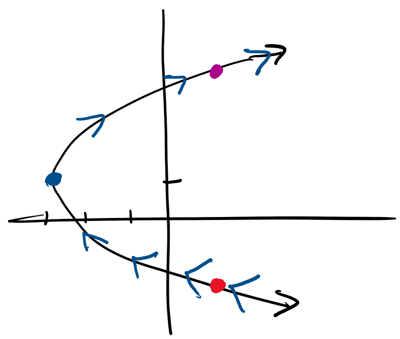
$$4t^2 = x + 3$$
$$t^2 = \frac{x+3}{4}$$
$$t = \pm\sqrt{\frac{x+3}{4}}$$

$$2t = y - 1$$
$$t = \frac{y-1}{2}$$

$$\boxed{x = 4\left(\frac{y-1}{2}\right)^2 - 3}$$

$$y = 4\left(\frac{x-1}{2}\right)^2 - 3$$





t	x	y	
0	-3	1	(-3,1)
1	1	3	(1,3)
-1	1	-1	

6. Find a vector equation of the line parallel to y=-6x+7 that passes through the point (-24,601).

Point: $\langle -24, 601 \rangle$

$$m = \frac{-6}{1} = \frac{\Delta y}{\Delta x}$$

$$\vec{m} = \langle \Delta x, \Delta y \rangle = \langle 1, -6 \rangle$$

or $\langle -1, 6 \rangle$

$$\vec{r}(t) = \vec{r_0} + t\,\vec{m}$$

$$\boxed{\begin{aligned}\vec{r}(t) &= \langle -24, 601 \rangle + t\langle 1, -6 \rangle \\ &= \langle -24 + t, 601 - 6t \rangle\end{aligned}}$$

$$\underbrace{x = -24 + t, \quad y = 601 - 6t}$$ Parametric Equations

7. Examine the following graph.

(a) Determine the locations of all discontinuities on the following graph. Explain why the function is discontinuous at each.

$x=-6$ $\quad f(-6)$ DNE

$x=-4$ $\quad f(-4)$ DNE

$x=0$ $\quad \lim_{x\to 0} f(x) \neq f(0)$

$x=2$ $\quad \lim_{x\to 2} f(x)$ DNE

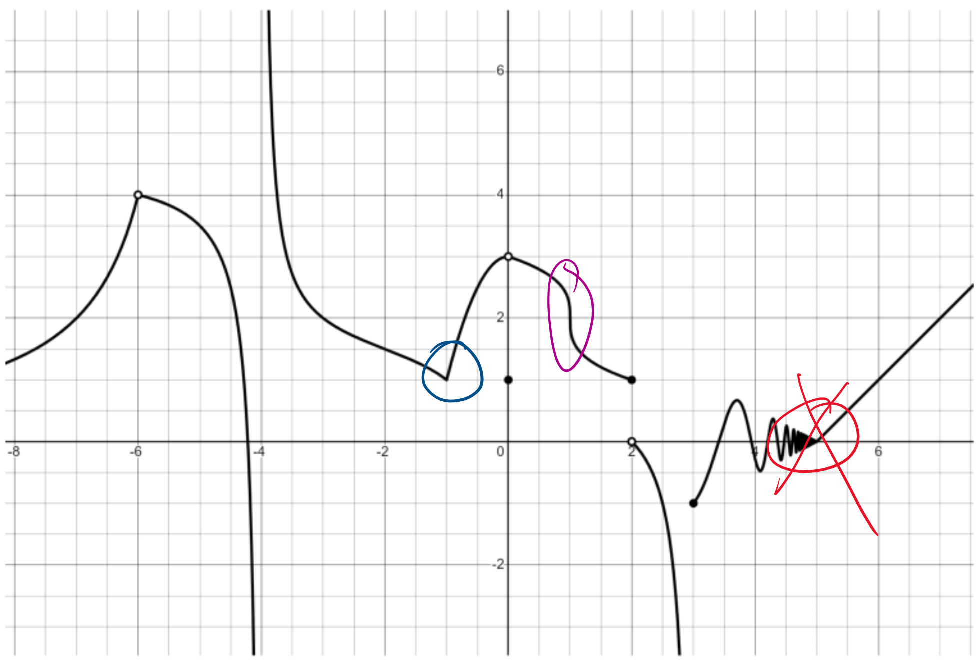
$x=3$ $\quad \lim_{x\to 3} f(x)$ DNE

(b) Determine the locations of each location where the function below is not differentiable. Explain why the derivative does not exist at each.

$x=-6, -4, 0, 2, 3$ Discontinuities

$x=-1$ Corner/Cusp

$x=1$ Vertical Tangent Line



* Typo

8. Evaluate the following limits.

(a) $\lim\limits_{x\to 2} \dfrac{4x-8}{x^2-4x+4} = \lim\limits_{x\to 2} \dfrac{4(x-2)}{(x-2)^2} = \lim\limits_{x\to 2} \dfrac{4}{x-2}$ $\boxed{\text{DNE}}$

$\dfrac{4(2)-8}{4-4(2)+4} = \dfrac{0}{0}$

$\dfrac{4}{2-2} = \dfrac{4}{0} \leftarrow \dfrac{\text{Nonzero}}{0} = \infty$ or $-\infty$

$\lim\limits_{x\to 2^-} f(x) = -\infty$

$\lim\limits_{x\to 2^+} f(x) = \infty$

$x = 1.9 \quad \dfrac{+}{1.9-2} = \dfrac{+}{-} \rightarrow -\infty$

$x = 2.1 \quad \dfrac{+}{2.1-2} = \dfrac{+}{+} \rightarrow +\infty$

(b) $\lim\limits_{t\to 0} \left(\dfrac{5}{t} - \dfrac{5}{t^2+t}\right) = \lim\limits_{t\to 0} \dfrac{(t+1)}{(t+1)}\dfrac{5}{t} - \dfrac{5}{t(t+1)}$

$\dfrac{5}{0} - \dfrac{5}{0}$

$= \lim\limits_{t\to 0} \dfrac{5t+5-5}{t(t+1)}$

$= \lim\limits_{t\to 0} \dfrac{5t}{t(t+1)} = \lim\limits_{t\to 0} \dfrac{5}{t+1} = \dfrac{5}{0+1} = \boxed{5}$

9. Determine the values of a and b that make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3} & \text{if} \quad x < -3 \\ ax^2+bx-3 & \text{if} \quad -3 \le x < 2 \\ 6x-b & \text{if} \quad x \le 2 \end{cases}$$

$\frac{x^2-9}{x+3}$ $\quad ax^2+bx-3 \quad 6x-b$

$-3 \quad 2$

~~$x \neq -3$~~

Not in interval

$\underline{x=-3}$

$f(-3) = a(3)^2 + b(-3) - 3$

$= 9a - 3b - 3$

$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{x^2-9}{x+3}$

$= \lim_{x\to -3^-} \frac{(x+3)(x-3)}{(x+3)} = \lim_{x\to -3^-} x-3$

$= -3-3 = -6$

Must be Equal

$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} ax^2+bx-3$

$= 9a - 3b - 3$

$9a - 3b - 3 = -6$

$9a - 3b = -3$

$4\left(\frac{12}{13}\right) + 3b = 15$

$3b = 15 - \frac{48}{13}$

$\underline{x=2}$

$f(2) = 6(2) - b = 12 - b$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2+bx-3$

$= 4a + 2b - 3$

Must be equal

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 6x - b = 12 - b$

$4a + 2b - 3 = 12 - b$

$4a + 3b = 15$

$+\ 9a - 3b = -3$

$13a = 12$

$\boxed{a = \frac{12}{13}}$

$\boxed{b = \frac{15 - 48/13}{3}}$

10. Consider the function $f(x) = \dfrac{4x^2+4x-8}{x^2-x-6}$.

(a) For what values of x is $f(x)$ undefined?

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$\boxed{x = 3 \quad x = -2}$

$f(3)$ DNE

$f(-2)$ DNE

(b) Determine the location ($x-$value) of any holes in the function. For each hole, what could we define $f(a)$ to be so that the function is continuous at a?

$$\lim_{x\to 3} \frac{4(x^2+x-2)}{(x-3)(x+2)}$$

$$\frac{4(9+3-2)}{0} = \frac{\text{Nonzero}}{0}$$

∞ or $-\infty$

$$\lim_{x\to -2} \frac{4(x+2)(x-1)}{(x-3)(x+2)} = \lim_{x\to -2} \frac{4(x-1)}{x-3} = \frac{4(-2-1)}{-2-3}$$

$$\frac{0}{0} \qquad = \frac{4(-3)}{-5} = \frac{-12}{-5} = \frac{12}{5}$$

Hole @ $\left(-2, \frac{12}{5}\right)$

(c) Determine any vertical asymptotes of the function. What is $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ for each?

$x = 3$ is a V.A.

(d) Determine any horizontal asymptotes of the function.

$$\lim_{x\to\infty} \frac{4x^2+4x-8}{1x^2-x-6} = \lim_{x\to\infty} \frac{x^2\left(4+\frac{4}{x}-\frac{8}{x^2}\right)}{x^2\left(1-\frac{1}{x}-\frac{6}{x^2}\right)}$$

$$\lim_{x\to\infty} \frac{4+\cancel{\frac{4}{x}}^{0}-\cancel{\frac{8}{x^2}}^{0}}{1-\cancel{\frac{1}{x}}^{0}-\cancel{\frac{6}{x^2}}^{0}} = \boxed{4}$$

$\boxed{\text{H.A. } y = 4}$

11. Determine the horizontal asymptotes for each of the following. If there is not a horizontal asymptote, determine $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

(a) $f(x) = \dfrac{888x^4 - 555x^5}{2x^2 + x^6}$

$$= \frac{x^5\left(\frac{888}{x} - 555\right)}{x^6\left(\frac{2}{x^4}+1\right)} = \frac{\frac{888}{x} - 555}{x\left(\frac{2}{x^4}+1\right)}$$

$\boxed{\text{H.A. } y = 0}$

$$\frac{-555}{x \cdot 1} = 0$$

$$\lim_{x\to\infty} \frac{-2e^{6x} + 7e^{-2x}}{11e^{6x} - e^{-3x}} = \lim_{x\to\infty} \frac{e^{6x}\left(-2+7e^{-8x}\right)}{e^{6x}\left(11 - e^{-9x}\right)}$$

$$= \lim_{x\to\infty} \frac{-2+7e^{-8x}}{11 - e^{-9x}} = \frac{-2+0}{11+0} = \frac{-2}{11}$$

(b) $f(x) = \dfrac{-2e^{6x} + 7e^{-2x}}{11e^{6x} - e^{-3x}}$

$\boxed{\text{H.A. } y = \frac{-2}{11}}$

$$\lim_{x\to-\infty} \frac{-2e^{6x} + 7e^{-2x}}{11e^{6x} - e^{-3x}} = \lim_{x\to-\infty} \frac{e^{-2x}\left(-2e^{8x}+7\right)}{e^{-3x}\left(11e^{9x} - 1\right)}$$

$$= \lim_{x\to-\infty} \frac{e^{x}\left(-2e^{8x}+7\right)}{11e^{9x}-1} = \frac{0\,(0+7)}{0-1} = \frac{0}{-1} = 0$$

$\boxed{\text{H.A. } y = 0}$

(c) $f(x) = \dfrac{3x - 5x^7}{2x^4 - 5}$

$\boxed{\text{No H.A.}}$

$$\lim_{x\to\infty} \frac{-5x^7}{2x^4} = \lim_{x\to\infty} \frac{-5x^3}{2} = -\infty$$

$$\lim_{x\to-\infty} \frac{-5x^7}{2x^4} = \lim_{x\to-\infty} \frac{-5x^3}{2} = \infty$$

$$\lim_{x\to\infty} \frac{\sqrt{4x^2-16} - 3x}{7x-2} = \lim_{x\to\infty} \frac{\sqrt{x^2(4-16/x^2)} - 3x}{7x-2}$$

$\boxed{\text{H.A. } y = \frac{-1}{7}}$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{4-16/x^2} - 3x}{7x-2}$$

(d) $f(x) = \dfrac{\sqrt{4x^2-16} - 3x}{7x-2}$

$$= \lim_{x\to\infty} \frac{x\sqrt{4-16/x^2} - 3x}{7x-2}$$

$$= \lim_{x\to\infty} \frac{x\left(\sqrt{4-16/x^2} - 3\right)}{x\left(7-2/x\right)} = \lim_{x\to\infty} \frac{\sqrt{4-16/x^2} - 3}{7 - 2/x} = \frac{\sqrt{4} - 3}{7} = \boxed{\frac{-1}{7}}$$

$$= \lim_{x\to-\infty} \frac{-x\sqrt{4-16/x^2} - 3x}{7x-2} = \lim_{x\to-\infty} \frac{x\left(-\sqrt{4-16/x^2} - 3\right)}{x\left(7-2/x\right)}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{4-16/x^2} - 3}{7 - 2/x} = \frac{-\sqrt{4} - 3}{7} = \boxed{\frac{-5}{7}}$$

$\boxed{\text{H.A. } y = \frac{-5}{7}}$

12. The position of an object on the $x-$axis is $x(t) = \frac{5}{t+1}$, where $t \geq 0$.

(a) Determine the average velocity of the object on the interval from $t = 0$ to $t = 4$.

Slope of Secant Line

Avg. Velocity $\frac{x(4) - x(0)}{4 - 0}$

$x(4) = \frac{5}{4+1} = 1$

$x(0) = \frac{5}{0+1} = 5$

$\frac{1-5}{4-0} = \frac{-4}{4} = \boxed{-1}$

$\lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{5}{(4+h)+1} - 1}{h}$

(b) Determine the instantaneous velocity of the object when $t = 4$.

$= \lim_{h\to 0} \frac{\frac{5}{5+h} - 1}{h} = \lim_{h\to 0} \frac{\frac{5}{5+h} - \frac{5+h}{5+h}}{\frac{h}{1}}$

$= \lim_{h\to 0} \frac{5 - (5+h)}{5+h} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-h}{(5+h)h}$

$\lim_{h\to 0} \frac{-1}{5+h} = \frac{-1}{5+0} = \boxed{\frac{-1}{5}}$

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13. Determine the derivative of $f(x) = \sqrt{3x^2 \text{[redacted]} + 5}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3(x+h)^2+5} - \sqrt{3x^2+5}}{h} \cdot \frac{\left(\sqrt{3(x+h)^2+5} + \sqrt{3x^2+5}\right)}{\left(\sqrt{3(x+h)^2+5} + \sqrt{3x^2+5}\right)}$$

$$= \lim_{h \to 0} \frac{\left[3(x+h)^2+5\right] - \left[3x^2+5\right]}{h\left(\sqrt{3(x+h)^2+5} + \sqrt{3x^2+5}\right)} = \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 5) - (3x^2+5)}{h\ (\text{Conjugate})}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h\ (\text{Conj})} = \lim_{h \to 0} \frac{\cancel{h}(6x+3h)}{\cancel{h}(\text{Conj})}$$

$$= \lim_{h \to 0} \frac{6x + 3h}{\sqrt{3(x+h)^2+5} + \sqrt{3x^2+5}} = \frac{6x + 0}{\sqrt{3(x+0)^2+5} + \sqrt{3x^2+5}}$$

$$= \boxed{\frac{6x}{2\sqrt{3x^2+5}}}$$

14. Which theorem from this course would help determine a solution to the following questions? If there is time, we will also solve these problems.

(a) Determine which of the following intervals contain at least one solution to the equation $12x - e^x = 5x^2 - 17$. Select all that apply.

- $(-2, -1)$
- $(-1, 0)$
- $(0, 1)$
- $(1, 2)$
- $(2, 3)$

Intermediate Value Theorem

$$\underbrace{12x - e^x - 5x^2 + 17}_{f(x)} = 0$$

DNE

(b) Evaluate $\displaystyle\lim_{x\to 5}\left((x-5)\sin\left(\frac{1}{x-5}\right)\right)$.

Squeeze Theorem

$$-1 \le \sin\left(\frac{1}{x-5}\right) \le 1$$