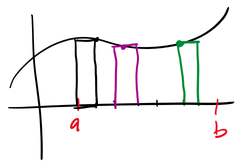
Week 13

Monday, November 18, 2024 7:25 PM

Math 151
Week-In-Review 13
5.1, 5.2, 5.3
Todd Schrader



Problem Statements

1. Suppose the area under the curve of a function is estimated using a Riemann Sum with n equal-width rectangles. $[a, b]$

(a) Determine a general formula for the sample points, x_i^*, using right endpoints.

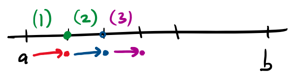
$\Delta x = \frac{b-a}{n}$

$\boxed{x_i^* = a + i \cdot \Delta x}$

$x_1^* = a + \Delta x$

$x_2^* = a + 2\Delta x$

$x_3^* = a + 3\Delta x$



(b) Determine a general formula for the sample points, x_i^*, using left endpoints.

$\Delta x = \frac{b-a}{n}$

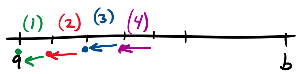
$\boxed{x_i^* = a + (i-1)\Delta x}$

$x_1^* = a$

$x_2^* = a + \Delta x$

$x_3^* = a + 2\Delta x$

$x_4^* = a + 3\Delta x$



(c) Determine a general formula for the sample points, x_i^*, using midpoints.

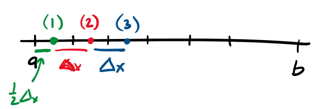
$\boxed{x_i^* = a + (i - 0.5)\Delta x}$

$x_1^* = a + 0.5\Delta x$

$x_2^* = a + 1.5\Delta x$

$x_3^* = a + 2.5\Delta x$

$x_4^* = a + 3.5\Delta x$



(d) Which of these formulas seems to be the most simple to work with?

Right Endpoints

2. Use this information to answer part (b) of the following.

Typo * Typo *

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Set up a limit to evaluate the exact area under the curve of $f(x) = -2 + 2x$ on the interval from $x = 1$ to $x = 4$.

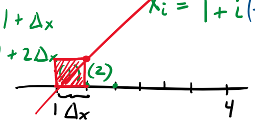
$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

(Right Endpoints)

$x_1^* = 1 + \Delta x$

$x_2^* = 1 + 2\Delta x$

$x_i^* = 1 + i\left(\frac{3}{n}\right)$



$$A_i = \frac{3}{n} \cdot f\left(1 + i \cdot \frac{3}{n}\right) = \frac{3}{n} \cdot \left[-2 + 2\left(1 + \frac{3i}{n}\right)\right]$$

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \cdot \left[-2 + 2\left(1 + \frac{3i}{n}\right)\right]}$$

(b) Use the above information to evaluate the limit.

$$A = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{6i}{n} = \lim_{n\to\infty} \frac{3}{n} \cdot \frac{6}{n} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{18}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{18n^2 + 18n}{2n^2} = \boxed{9}$$

Equal

(c) Write the limit as an equivalent definite integral.

$f(x) = -2 + 2x \qquad 1 \le x \le 4$

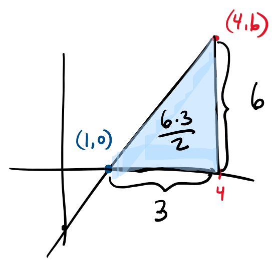
$$\boxed{A = \int_1^4 (-2 + 2x)\, dx}$$

(d) Evaluate the integral using geometry.

$f(1) = -2 + 2(1) = 0$

$f(4) = -2 + 2(4) = 6$

$$\boxed{A = 9}$$



$$\sum_{i=1}^{7} i = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

$$= 28$$

$$= \frac{7(7+1)}{2} = \frac{7(8)}{2} = 7 \cdot 4 = 28$$

$$\frac{3}{n} \sum \left(1 + \frac{6i}{n}\right)$$

$$\frac{3}{n}\left[\sum_{i=1}^{n} 1 + \frac{6}{n}\sum i\right]$$

$$\sum_{i=1}^{7} 1 = \underbrace{1+1+1+1+1+1+1}_{=7}$$

Typo*

3. Use this information to answer part (b) of the following.

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Set up a limit to evaluate the exact area under the curve of $f(x) = x^2$ on the interval $[0, 4]$.

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n} \qquad x_i^* = a + i\Delta x = 0 + i\left(\frac{4}{n}\right) = \frac{4i}{n}$$

$$A_i = \frac{4}{n} \cdot f\left(\frac{4i}{n}\right) = \frac{4}{n}\left(\frac{4i}{n}\right)^2$$

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \frac{16 i^2}{n^2}}$$

(b) Use the above information to evaluate the limit.

$$A = \lim_{n\to\infty} \frac{4}{n} \cdot \frac{16}{n^2} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{64 \cdot 2}{6} = \boxed{\frac{64}{3}}$$

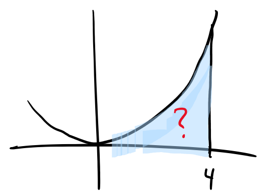
Equal

(c) Write the limit as an equivalent definite integral.

$$\boxed{A = \int_0^4 x^2 \, dx}$$

(d) Can you evaluate the integral using geometry?

Not Easily

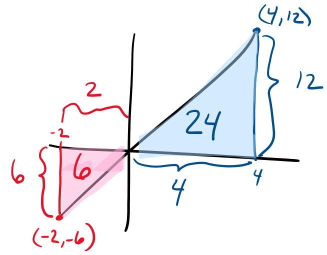


4. Evaluate the following integrals geometrically.

(a) $\int_{-2}^{4} 3x\, dx$

$f(-2) = 3(-2) = -6$

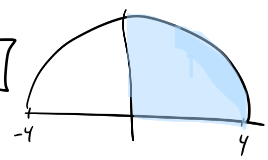
$f(4) = 3(4) = 12$



$$\int_{-2}^{4} 3x\, dx = -6 + 24 = \boxed{18}$$

Quarter Circle

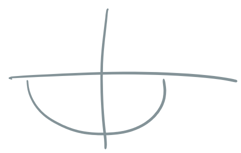
(b) $\int_{0}^{4} \sqrt{16 - x^2}\, dx = \dfrac{\pi(4)^2}{4} = \boxed{4\pi}$



$y = \sqrt{16 - x^2}$

$y^2 = 16 - x^2$

$x^2 + y^2 = 16$

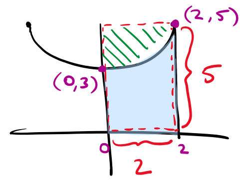


$5 + (-\sqrt{4 - x^2})$

(c) $\int_{0}^{2} 5 - \sqrt{4 - x^2}\, dx = \int_0^2 5\, dx - \int_0^2 \sqrt{4 - x^2}\, dx$

$f(2) = 5 - \sqrt{4 - (2)^2} = 5$

$f(0) = 5 - \sqrt{4 - 0} = 3$



$$\int_0^2 5 - \sqrt{4 - x^2}\, dx = 5(2) - \frac{\pi(2)^2}{4}$$

$$= \boxed{10 - \pi}$$

$\Delta x = \frac{b-a}{n} = \frac{3-(-1)}{n} = \frac{4}{n}$

$\Delta x = \frac{-1-3}{n} = \frac{-4}{n}$

5. Suppose $\int_{-1}^{3} f(x)\,dx = 5$, $\int_{-1}^{3} g(x)\,dx = 3$ and $\int_{-1}^{3} h(x)\,dx = -7$. Evaluate the following.

(a) $\int_{-1}^{3} 4f(x)\,dx + \int_{3}^{-1} (g(x) - h(x))\,dx$

$$= 4\int_{-1}^{3} f(x)\,dx - \int_{-1}^{3} (g(x) - h(x))\,dx$$

$$= 4(5) - \left[\int_{-1}^{3} g(x)\,dx - \int_{-1}^{3} h(x)\,dx\right]$$

$$= 4(5) - [(3) - (-7)]$$

$$= 20 - 10 = \boxed{10}$$

$\Delta x = \frac{9-9}{n} = \frac{0}{n} = 0$

(b) $\underbrace{\int_{-4}^{0} f(x)\,dx + \int_{-1}^{-4} f(x)\,dx}$ $- \int_{3}^{0} f(x)\,dx + \underbrace{\int_{9}^{9} f(x)\,dx}_{0} + \int_{2}^{5} 3\,dx$

3

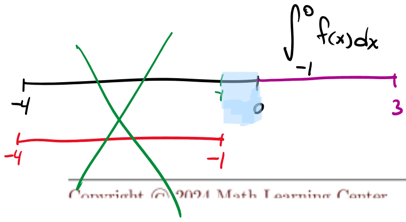
$\int_{-1}^{-4} f(x)\,dx + \int_{-4}^{0} f(x)\,dx$

$+\int_{0}^{3} f(x)\,dx$

$\int_{-1}^{0} f(x)\,dx$

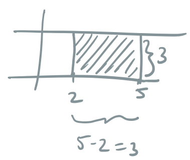
$\int_{-4}^{0} f(x)\,dx - \int_{-4}^{-1} f(x)\,dx + \int_{0}^{3} f(x)\,dx$

$\int_{-1}^{0} f(x)dx$



$$\int_{-1}^{3} f(x)\,dx + \int_{2}^{5} 3\,dx$$

$$5 + 9 = \boxed{14}$$



$$\int_a^b f(x)\,dx = F(b) - F(a)$$

6. Evaluate the integrals, if possible.

(a) $\int_0^4 x^2\,dx = \left[\frac{1}{3}x^3 + C\right]_0^4$

$$= \left[\frac{1}{3}(4)^3 + C\right] - \left[\frac{1}{3}(0)^3 + C\right]$$

$$= \frac{64}{3} + C - 0 - C = \boxed{\frac{64}{3}}$$

(b) $\int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_1^9$

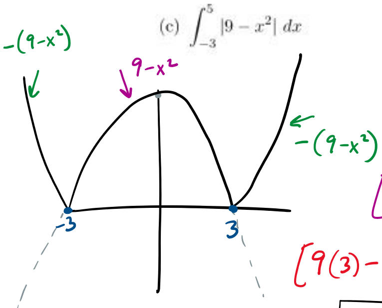
$$= \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2}$$

$$= \frac{2}{3}(27) - \frac{2}{3}(1) = \frac{2}{3}(26) = \boxed{\frac{52}{3}}$$

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$|9-x^2| = \begin{cases} 9-x^2 & \text{if } 9-x^2 \ge 0 \\ -(9-x^2) & \text{if } 9-x^2 < 0 \end{cases}$$

(c) $\int_{-3}^5 |9-x^2|\,dx$



$9-x^2 = 0 \qquad x = \pm 3$

$$\int_{-3}^5 |9-x^2|\,dx = \int_{-3}^3 9-x^2\,dx + \int_3^5 -(9-x^2)\,dx$$

$$\left[9x - \frac{1}{3}x^3\right]_{-3}^3 + \left[-9x + \frac{1}{3}x^3\right]_3^5$$

$$\left[9(3) - \frac{1}{3}(27)\right] - \left[9(-3) - \frac{1}{3}(-27)\right] + \left[-9(5) + \frac{1}{3}(125)\right] - \left[-9(3) + \frac{1}{3}(27)\right]$$

$$18 - (-27+9) + \left(-45 + \frac{125}{3}\right) - (-27+9)$$

$\cancel{\int_0^1 x^e \, dx + \int_0^1 e^x \, dx}$

(d) $\int_0^1 (x^e + e^x + 3^x)\ dx$

$= \left[\frac{1}{e+1} x^{e+1} + e^x + \frac{3^x}{\ln(3)}\right]_0^1$

$= \left[\frac{1}{e+1}(1)^{e+1} + e^1 + \frac{3^1}{\ln(3)}\right] - \left[\frac{1}{e+1}(0)^{e+1} + e^0 + \frac{3^0}{\ln(3)}\right]$

$= \boxed{\frac{1}{e+1} + e + \frac{3}{\ln(3)} - 1 - \frac{1}{\ln(3)}}$

(e) $\int_0^{\pi/3} \sec(x)\tan(x)\ dx$

$= \sec(x)\Big|_0^{\pi/3}$

$= \sec\left(\frac{\pi}{3}\right) - \sec(0)$

$= \frac{1}{(1/2)} - \frac{1}{(1)} = 2 - 1 = \boxed{1}$

(f) $\int_0^{\pi} \sec^2(x)\ dx$?

$\int_0^{\pi} \frac{1}{\cos^2(x)}\ dx$

$\cos^2(x) = 0$

$\cos(x) = 0$

$x = \frac{\pi}{2}$

$\cancel{= \tan(x)\Big|_0^{\pi}}$

$\cancel{= \tan(\pi) - \tan(0) = 0 - 0 = 0}$

$\frac{\pi}{2}$ is not in Domain of $f(x)$.

7. Find the derivative of the following functions.

(a) $g(x) = \int_0^x \underbrace{\sin\left(\frac{\pi t^2}{2}\right)}_{f(t)} \, dt = F(t)\Big|_0^x = F(x) - F(0)$

$g'(x) = \frac{d}{dx}\left[F(x) - F(0)\right] = f(x) - 0$

$\boxed{g'(x) = \sin\left(\frac{\pi x^2}{2}\right)}$

$\int_0^x$

(b) $h(w) = \int_w^{0} e^{x^2} \, dx = -\int_0^w e^{x^2} \, dx$

$h(w) = -\left(F(w) - F(0)\right)$

$h'(w) = -\left(f(w) - 0\right)$

$\boxed{h'(w) = -e^{w^2}}$

(c) $F(y) = \int_{-1}^{y} \frac{1}{1+t^3} \, dt = \int_{-1}^{0} \frac{1}{1+t^3} \, dt + \int_0^y \frac{1}{1+t^3} \, dt$

$F'(y) = \frac{1}{1+y^3}$

8. Find the derivative of the following functions.

(a) $g(x) = \int_0^{x^2} \underbrace{t\cos(t)}_{f(t)}\, dt = F(t)\Big|_0^{x^2} = F(x^2) - F(0)$

$g'(x) = \frac{d}{dx}\left[F(x^2) - F(0)\right] = f(x^2)\cdot 2x - \cancel{f(0)\cdot 0}$

$$\boxed{g'(x) = x^2\cos(x^2)\cdot 2x}$$

(b) $F(r) = \int_r^{3r} \arctan(t)\, dt$

$$\boxed{F'(r) = \arctan(3r)\cdot 3 - \arctan(r)\cdot 1}$$

(c) $g(x) = \int_{\sin(x)}^{\cos(x)} \ln(1+5\theta)\, d\theta$

$$\boxed{g'(x) = \ln(1+5\cos x)\cdot(-\sin x) - \ln(1+5\sin x)\cdot(\cos x)}$$