Math 151 - Week-In-Review 14

Topics for the week:

5.4 Indefinite Integrals and the Net Change Theorem
5.5 The Substitution Rule
Some Exam 2 Reminders

5.4 Indefinite Integrals and the Net Change Theorem

1. Determine the general indefinite integral $\int \left(\sqrt[5]{x^2} - \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x}} \right) dx$.

$$\int \Big(\underbrace{x^{2/5} - \frac{1}{\sqrt{1-x^2}} + x^{-1/2}}_{\text{derivative function}} \Big) dx = \underbrace{\frac{5}{7} x^{7/5} - \arcsin(x) + 2x^{1/2} + C}_{\text{anti derivative function}}$$

$$\text{Check: } \frac{d}{dx}\left[\frac{5}{7} x^{7/5} - \arcsin(x) + 2x^{1/2} + C \right] = \frac{5}{7} \cdot \frac{7}{5} x^{2/5} - \frac{1}{\sqrt{1-x^2}} + 2 \cdot \frac{1}{2} x^{-1/2} + 0$$

$$= x^{2/5} - \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x}} \checkmark$$

2. Determine the general indefinite integral $\int \left(\sec^2(t) - 2e^t + \frac{6}{t} \right) dt$.

$$\int \left(\sec^2(t) - 2e^t + \frac{6}{t} \right) dt = \tan(t) - 2e^t + 6 \cdot \ln|t| + C$$

3. Determine the general indefinite integral $\int \left(\frac{\sin(2x)}{\cos(x)}\right)\, dx$.

$$\int\left(\frac{\sin(2x)}{\cos(x)}\right)dx = \int\left(\frac{2\sin(x)\cos(x)}{\cos(x)}\right)dx$$

$$= \int(2\sin(x))\,dx \quad \text{or} \quad 2\int \sin(x)\,dx$$

$$= 2(-\cos(x)) + C$$

$$= -2\cos(x) + C$$

4. Determine the general indefinite integral $\int ((1-4t)(2t+6))\, dt$.

$$\int((1-4t)(2t+6))\,dt = \int(2t+6-8t^2-24t)\,dt$$

$$= \int(-8t^2-22t+6)\,dt$$

$$= -\frac{8}{3}t^3 - \frac{22}{2}t^2 + 6t + C$$

$$= -\frac{8}{3}t^3 - 11t^2 + 6t + C$$

5. Evaluate the definite integral $\int_2^\pi \left(x^2 - \cos(x) + \frac{1}{x}\right)\, dx$.

$$\int_{x=2}^{x=\pi} \left(x^2 - \cos(x) + \frac{1}{x}\right) dx = \left(\frac{1}{3}x^3 - \sin(x) + \ln|x| + C\right)\Bigg|_{x=2}^{x=\pi}$$

$$= \left[\frac{1}{3}(\pi)^3 - \sin(\pi) + \ln|\pi| + C\right] - \left[\frac{1}{3}(2)^3 - \sin(2) + \ln|2| + C\right]$$

$$= \frac{\pi^3}{3} - (0) + \ln(\pi) \underbrace{+ C} - \frac{8}{3} + \sin(2) - \ln(2) \underbrace{- C}$$

$$= \frac{\pi^3}{3} + \ln(\pi) - \frac{8}{3} + \sin(2) - \ln(2)$$

6. Evaluate the definite integral $\int_0^3 \left(-e^x + \sqrt{\frac{x}{8}}\right)\, dx$.

$$\int_{x=0}^{x=3} \left(-e^x + \sqrt{\frac{x}{8}}\right) dx = \int_{x=0}^{x=3} \left(-e^x + \frac{\sqrt{x}}{\sqrt{8}}\right) dx$$

$$= \int_0^3 \left(-e^x + \frac{1}{\sqrt{8}} x^{1/2}\right) dx$$

$$= \left[-e^x + \frac{2}{3\sqrt{8}} x^{3/2}\right]\Bigg|_{x=0}^{x=3}$$

$$= \left[-e^{(3)} + \frac{2}{3\sqrt{8}}(3)^{3/2}\right] - \left[-e^{(0)} + \frac{2}{3\sqrt{8}}(0)^{3/2}\right]$$

$$= -e^3 + \frac{2}{3 \cdot 2\sqrt{2}} \cdot 3\sqrt{3} + \frac{e^0}{1} - 0$$

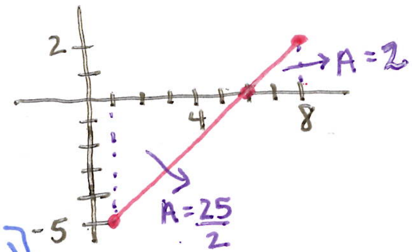
$$= -e^3 + \frac{\sqrt{3}}{\sqrt{2}} + 1$$

7. Explain why the Fundamental Theorem of Calculus Part 2 and the Net Change Theorem may not be applied to $\int_{-2}^{2}\left(5v^3+\frac{6}{v^5}\right)\,dv$. Is given an option for a new lower bound, a, such that the theorems would apply.

$f(v) = 5v^3 + \frac{6}{v^5}$ is not defined at $v=0$

options for a, $0 < a < 2$

8. A particle is moving in straight line motion that is expressed by the formula: $v(t) = t - 6$ (measured in meters per second).



a. Compute the displacement from $t = 1$ to $t = 8$.

$$\int_{t=1}^{t=8}(t-6)\,dt = \left(\frac{1}{2}t^2 - 6t\right)\Big|_{t=1}^{t=8}$$

$$= \left[\frac{1}{2}(8)^2 - 6(8)\right] - \left[\frac{1}{2}(1)^2 - 6(1)\right]$$

$$= \frac{64}{2} - 48 - \frac{1}{2} + 6$$

$$= 32 - 48 - \frac{1}{2} + 6$$

$$= -\frac{21}{2}$$

b. Compute the total distance traveled from $t = 1$ to $t = 8$.

$$\text{Distance} = -\int_1^6 v(t)\,dt + \int_6^8 v(t)\,dt$$

$$= -\left(-\frac{25}{2}\right) + (2)$$

$$= \frac{29}{2}$$

or $-\int_1^6 (t-6)\,dt + \int_6^8 (t-6)\,dt$

$$= \left(-\frac{1}{2}t^2 + 6t\right)\Big|_1^6 + \left(\frac{1}{2}t^2 - 6t\right)\Big|_6^8$$

$$= \left[-\frac{1}{2}(6)^2 + 6(6)\right] - \left[-\frac{1}{2}(1)^2 + 6(1)\right] + \left[\frac{1}{2}(8)^2 - 6(8)\right]$$

$$- \left[\frac{1}{2}(6)^2 - 6(6)\right] = \frac{29}{2}$$

5.5 The Substitution Rule

9. Determine $\int 5x^2\sqrt{x^3-7}\,dx$.

$u = x^3 - 7$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$\int \left(5x^2\sqrt{\underbrace{x^3-7}_{u}}\right)dx = \int 5\sqrt{u}\left(\frac{1}{3}du\right)$$

$$= \int \left(\frac{5}{3}u^{1/2}\right)du$$

$$= \frac{5}{3}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{10}{9}\left(x^3-7\right)^{3/2} + C$$

10. Calculate $\int \tan(3x)\,dx$.

$u = \cos(3x)$

$du = -3\sin(3x)\,dx$

$-\frac{1}{3}du = \sin(3x)\,dx$

$$\int (\tan(3x))\,dx = \int \left(\frac{\sin(3x)}{\underbrace{\cos(3x)}_{u}}\right)dx$$

$$= \int \left(\frac{1}{u}\right)\left(-\frac{1}{3}du\right)$$

$$= -\frac{1}{3}\ln|u| + C$$

$$= -\frac{1}{3}\ln|\cos(3x)| + C$$

11. Find $\int (2-x)e^{8x-2x^2}\,dx$.

$$\int \Big[\underbrace{(2-x)\,dx}_{\frac{1}{4}du}\; e^{\overbrace{8x-2x^2}^{u}}\Big] = \int \left(e^{u}\right)\left(\tfrac{1}{4}du\right)$$

$$= \frac{1}{4}e^{u} + C$$

$$= \frac{1}{4}e^{8x-x^2} + C$$

$u = 8x-2x^2$

$du = (8-4x)\,dx$

$du = 4(2-x)\,dx$

$\frac{1}{4}du = (2-x)\,dx$

12. Determine the general indefinite integral $\int \frac{1}{17x+90}\,dx$.

$$\int \left(\frac{1}{\underbrace{17x+90}_{u}}\right) \overbrace{dx}^{\frac{1}{17}du} = \int \left(\frac{1}{u}\right)\left(\frac{1}{17}du\right)$$

$$= \frac{1}{17}\ln|u| + C$$

$$= \frac{1}{17}\ln|17x+90| + C$$

$u = 17x+90$

$du = 17\,dx$

$\frac{1}{17}du = dx$

13. Evaluate $\displaystyle\int_1^{e^3} \frac{\ln^2(x)}{x}\,dx$.

$$\int_{x=1}^{x=e^3} \Big(\underbrace{\frac{\ln^2(x)}{x}\,dx}_{du}\Big) = \int_{u=0}^{u=3} \big(u^2 \big)\,du \qquad \ln(x)=u$$

$u = \ln(x)$

$du = \frac{1}{x}dx$

$x=1 \Rightarrow \ln(1) = 0 = u$

$x=e^3 \Rightarrow \ln(e^3) = 3 = u$

$$= \left(\frac{1}{3}u^3\right)\Big|_{u=0}^{u=3}$$

$$= \frac{1}{3}(3)^3 - \frac{1}{3}(0)^3$$

$$= 9$$

14. Evaluate $\displaystyle\int_4^{9} \frac{x}{x-3}\,dx$.

$$\int_{x=4}^{x=9} \Big(\frac{x}{\underbrace{x-3}_{u}}\Big)\,dx = \int_{u=1}^{u=6} \Big(\frac{u+3}{u}\Big)\,du$$

$u = x-3$ and $u+3 = x$

$du = 1\,dx$

$x=9 \Rightarrow 9-3 = 6 = u$

$x=4 \Rightarrow 4-3 = 1 = u$

$$= \int_{u=1}^{u=6} \Big(1 + \frac{3}{u}\Big)\,du$$

$$= \big(u + 3\ln|u|\big)\Big|_{u=1}^{u=6}$$

$$= \big[6 + 3\ln|6|\big] - \big[1 + 3\ln|1|\big]$$

$$= 6 + 3\ln|6| - 1 - 0$$

$$= 5 + 3\ln|6|$$

E2 Some Reminders from Exam 2

15. Given the function $f(x) = \dfrac{x}{x^2+1}$,

(a) Determine the equation of the line tangent to the curve at $x = 3$.

$f(3) = \dfrac{3}{9+1} = \dfrac{3}{10}$

$y - y_1 = m(x - x_1)$

$\dfrac{df(x)}{dx} = \dfrac{1(x^2+1) - 2x(x)}{(x^2+1)^2}$

$y - \dfrac{3}{10} = \dfrac{-2}{25}(x-3)$

$\left.\dfrac{df(x)}{dx}\right|_{x=3} = \dfrac{(9+1) - 2(9)}{(9+1)^2} = \dfrac{-8}{100} = \dfrac{-2}{25}$

(b) Determine any values of x for which the tangent line to the curve is horizontal.

$\dfrac{df(x)}{dx} = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

$\dfrac{df(x)}{dx} = 0$

$\dfrac{-x^2+1}{(x^2+1)^2} = 0$

$\boxed{\text{at } x = -1 \text{ or } x = 1}$

$-x^2+1=0$

$x^2 = 1$

$x = \pm 1$

16. For $f(x) = \log(x) + 3^{x^2}$, compute $f''(1)$.

$f(x) = \log(x) + 3^{x^2}$

$f'(x) = \dfrac{1}{x\ln(10)} + 3^{x^2} \cdot \ln(3) \cdot 2x$

$f''(x) = \dfrac{-1}{x^2\ln(10)} + 2 \cdot 3^{x^2} \ln(3) + 3^{x^2} \cdot \ln(3) \cdot 2x \cdot \ln(3) \cdot 2x$

$f''(x) = \dfrac{-1}{x^2\ln(10)} + 3^{x^2}\left[2\ln(3) + 4x^2\ln^2(3)\right]$

$f''(1) = \dfrac{-1}{1\ln(10)} + 3^1\left[2\ln(3) + 4\ln^2(3)\right]$

17. Compute $\frac{dy}{dx}$ for $y\sin(x^2) = x\sin(y^2)$.

$$y\sin(x^2) = x\sin(y^2)$$

$$\frac{dy}{dx}\sin(x^2) + 2x\cos(x^2)\,y = 1\sin(y^2) + 2y\cos(y^2)\frac{dy}{dx}\,x$$

$$\frac{dy}{dx}\sin(x^2) - 2xy\cos(y^2)\frac{dy}{dx} = \sin(y^2) - 2xy\cos(x^2)$$

$$\frac{dy}{dx}\left[\sin(x^2) - 2xy\cos(y^2)\right] = \sin(y^2) - 2xy\cos(x^2)$$

$$\frac{dy}{dx} = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}$$

18. Compute $\frac{dy}{dx}$ for $y = \left(1 + \frac{2}{x}\right)^{\frac{1}{x}}$.

$$y = \left(1 + \frac{2}{x}\right)^{1/x}$$

$$\ln(y) = \ln\left[\left(1 + 2/x\right)^{1/x}\right]$$

$$\ln(y) = \frac{1}{x}\cdot\ln\left(1 + \frac{2}{x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{x^2}\cdot\ln(1 + 2/x) + \frac{1}{1 + 2/x}\cdot\frac{-2}{x^2}\cdot\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-\ln(1 + 2/x)}{x^2} - \frac{2}{x^3 + 2x^2}$$

$$\frac{dy}{dx} = (1 + 2/x)^{1/x}\left[\frac{-\ln(1 + 2/x)}{x^2} - \frac{2}{x^3 + 2x^2}\right]$$

19. Compute $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dy}{dx}$ for $x = t^4 + 9t^2$ and $y = \arctan\left(t^2 + 64\right)$.

$x = t^4 + 9t^2$

$\frac{dx}{dt} = 4t^3 + 18t$

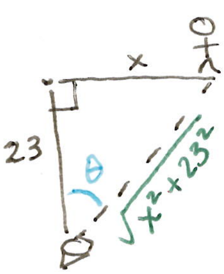
$y = \arctan(t^2 + 64)$

$\frac{dy}{dt} = \frac{2t}{1 + (t^2+64)^2}$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2t}{1+(t^2+64)^2}}{4t^3 + 18t}$$

$$\frac{dy}{dx} = \left[\frac{2t}{1+(t^2+64)^2}\right] \cdot \left[\frac{1}{4t^3+18t}\right] = \frac{1}{\left(1+(t^2+64)^2\right)\left(2t^2+9\right)}$$

20. An actor slowly walks along a straight path at the back of the stage at a speed of 2 ft/s. A spotlight located at the front of the stage (23 feet from the back of the stage) is focused on the actor as they move. At what rate is the spotlight rotating when the actor is 14 ft from the point at the back of the stage closest to the spotlight?



$x = 14$ ft

$\frac{dx}{dt} = 2$ ft/s

Note:
$\sec(\theta) = \frac{\sqrt{14^2+23^2}}{23}$

$\tan(\theta) = \frac{x}{23}$

$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{23} \cdot \frac{dx}{dt}$

$\left(\frac{\sqrt{14^2+23^2}}{23}\right)^2 \cdot \frac{d\theta}{dt} = \frac{1}{23} \cdot (2)$

$\frac{d\theta}{dt} = \frac{2}{23} \cdot \left(\frac{23^2}{14^2+23^2}\right)$ rad/s

$\frac{d\theta}{dt} = \frac{46}{725}$ rad/s