Math 152 - Exam 3 Review

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Review of Maclaurin series. (center is at $a=0$)

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $= 1 + x + x^2 + x^3 + \cdots$ $R=1$
2. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $R=\infty$
3. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $R=\infty$
4. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $R=\infty$
5. $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ $R=1$
6. $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ $R=1$

Evaluate the following integral as Power Series.

1. $f(x) = \int 5x^2 \arctan(7x^3)\, dx$

Since, $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)}$

then $\arctan(7x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (7x^3)^{2n+1}}{(2n+1)}$

$$f(x) = \int (5x^2) \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 7^{2n+1} \cdot (x^3)^{2n+1}}{(2n+1)}\, dx$$

$$= 5 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 7^{2n+1}}{(2n+1)} \int x^2 \cdot x^{6n+3}\, dx$$

$$\int x^{6n+5}\, dx$$

$$\int x^n\, dx = \frac{x^{n+1}}{(n+1)}$$

$$f(x) = 5 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 7^{2n+1}}{(2n+1)} \cdot \frac{x^{6n+6}}{(6n+6)} + C$$

$a=0 \longrightarrow$ Maclaurin series

2. Find a Power series representation of the functions $f(x) = \dfrac{x^2}{(5-3x)^2}$.

$f(x) = x^2 \cdot g(x)$ where $g(x) = \dfrac{1}{(5-3x)^2}$

$$\int g(x)\,dx = \int \frac{1}{(5-3x)^2}\,dx$$

$u = 5-3x \Rightarrow \int \frac{1}{u^2}\left(\frac{du}{-3}\right)$

$du = -3\,dx$

$\frac{du}{-3} = dx$

$= \left(-\frac{1}{3}\right)\left(-\frac{1}{u}\right)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{u}\right)$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{5-3x}\right)$$

$$= \left(\frac{1}{3}\right)\frac{1}{5\left(1-\frac{3x}{5}\right)}$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{5}\right)\left(\frac{1}{1-\frac{3x}{5}}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{5}\right)\sum_{n=0}^{\infty}\left(\frac{3x}{5}\right)^n$$

$$= \frac{1}{3}\cdot\frac{1}{5}\sum_{n=0}^{\infty}\frac{3^n \cdot x^n}{5^n} = \sum_{n=0}^{\infty}\frac{3^{n-1}\cdot x^n}{5^{n+1}} + C$$

$(n=0)$ term: $\frac{3^{-1}\cdot x^0}{5^1} = \frac{1}{3\cdot 5} = \frac{1}{15}$

$$g(x) = \frac{1}{(5-3x)^2} = \frac{d}{dx}\int g(x)\,dx = \sum_{n=0}^{\infty}\frac{3^{n-1}}{5^{n+1}}\cdot\frac{d}{dx}(x^n)$$

$$= \sum_{n=1}^{\infty}\frac{3^{n-1}}{5^{n+1}}\cdot n\cdot x^{n-1}$$

$n \rightarrow (n+1)$
$$= \sum_{n=0}^{\infty}\frac{3^n(n+1)\cdot x^n}{5^{n+2}}$$

$$\boxed{f(x) = x^2 \cdot g(x) = \sum_{n=0}^{\infty}\frac{3^n(n+1)}{5^{n+2}}\cdot x^{n+2}}$$

3. If $f(x)=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$, find the power series for $f'(x)$ (a) and $\int f(x)dx$ (b). Identify $f(x)$ (c).

$$f(x)=\sum_{n=0}^{\infty}\frac{3^n\cdot x^n}{n!}$$

$(n=0)$ term $=\frac{3^0x^0}{0!}=1$

a) $$f'(x)=\sum_{n=0}^{\infty}\frac{3^n}{n!}\frac{d}{dx}(x^n)=\sum_{n=1}^{\infty}\frac{3^n}{n!}n\cdot x^{n-1}=\sum_{n=0}^{\infty}\frac{3^{n+1}}{(n+1)!}(n+1)\cdot x^n$$

b) $$\int f(x)dx=\sum_{n=0}^{\infty}\frac{3^n}{n!}\int x^n dx=\sum_{n=0}^{\infty}\frac{3^n}{n!}\cdot\frac{x^{n+1}}{(n+1)}+c$$

c) $$f(x)=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}=e^{3x}$$

4. Find the 25th derivative for the function $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(n+2)}x^n$ centered at $x=0$.

$a=0$ → Maclaurin

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}\cdot x^n$$

$$n!\cdot\frac{(-1)^n}{3^n\cdot(n+2)}\cdot x^n=\frac{f^n(0)}{n!}x^n\cdot n!\Rightarrow f^n(0)=\frac{(-1)^n\cdot n!}{3^n(n+2)}$$

25th derivative → $n=25$

$$f^{25}(0)=\frac{(-1)^{25}(25)!}{3^{25}(25+2)}=-\frac{(25)!}{3^{25}(27)}$$

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$

Find the Taylor Series Representions for the following functions

5. $f(x) = xe^{3x}$ centered at $x = 5$ $\rightarrow$ $f(x) = \sum_{n=0}^{\infty} \frac{f^n(5)}{n!}(x-5)^n$

solve for $f^n(5)$

n	$f^n(x)$
0	xe^{3x} $\rightarrow f^0(5) = 5e^{15}$
1	$e^{3x} + 3xe^{3x} = e^{3x}(1+3x)$
2	$3e^{3x} + 3e^{3x} + 3x\,3e^{3x} = 3e^{3x}(1+1+3x) = 3e^{3x}(2+3x)$
3	$3\cdot3e^{3x} + 3\cdot3e^{3x} + 9e^{3x} + 9x\cdot3e^{3x} = 3^2e^{3x}(3+3x)$

$f^n(x) = 3^{n-1}\cdot e^{3x}(n+3x)$

$f^n(5) = 3^{n-1}\cdot e^{15}(n+15)$

$f(x) = \sum_{n=0}^{\infty} \frac{3^{n-1}\cdot e^{15}\cdot(n+15)}{n!}(x-5)^n$

$n=0$: $\frac{3^{-1}\cdot e^{15}(15)}{0!}(x-5)^0 = \frac{1}{3}e^{15}(15) = 5e^{15}$

6. $f(x) = \ln(1 + x)$ centered at $a = 2$

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(x-2)^n$, solve for $f^n(2)$

n	$f^n(x)$
0	$\ln(1+x)$ $\rightarrow f^0(2) = \ln(3)$
1	$\frac{1}{1+x} = (1+x)^{-1}$
2	$(-1)(1+x)^{-2}$
3	$(-1)(-2)(1+x)^{-3}$
4	$(-1)(-2)(-3)(1+x)^{-4}$ = $(-1)(-1)(-1)(1)(2)(3)$

$f^n(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$

$f^n(2) = (-1)^{n-1}(n-1)!\,(1+2)^{-n}$

$n \geq 1$: $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot(n-1)!}{3^n\cdot n!}(x-2)^n + \ln(3)$ ($n=0$)

$$f(x) = \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n\cdot n}(x-2)^n$$

e^{-x^2} $\quad \int e^{-x^2}\,dx$

$a=0$

Find the Maclaurin Series Representation for the following functions.

7. $f(x) = \int_0^x e^{-t^2}\,dt.$

$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot t^{2n}}{n!}$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \int_0^x t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{t^{2n+1}}{(2n+1)} \Bigg|_{t=0}^{t=x}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{(2n+1)} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{0^{2n+1}}{(2n+1)} \rightarrow 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{(2n+1)} = \int_0^x e^{-t^2}\,dt$$

8. $f(x) = x^3 \cos(2x)$

$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$

$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2x)^{2n}}{(2n)!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot x^{2n}}{(2n)!}$

$$f(x) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot x^{2n}}{(2n)!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n}}{(2n)!} \cdot x^{2n+3}$$

Find the sum of the following series.

9. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(3)^n(\pi^n)}{n!}$ $= \sum_{n=0}^{\infty} \frac{(-3\pi)^n}{n!} = e^{-3\pi}$

$\sim e^x$

10. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!}$ $= \sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n+1}}{(2n+1)!} = \sin(3) = 3 - \frac{3^3}{3!} + \frac{3^5}{5!} - \frac{3^7}{7!} + \cdots$

n=0 n=1 n=2 n=3 ...

new problem

$f(x) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{3^{2n+1}}{(2n+1)!} = \sin(3) - 3$

11. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n)!}$ $= \sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n}}{(2n)!} \cdot 3 = 3 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n}}{(2n)!}$

$3^{2n} \cdot 3$

$= 3\cos(3)$

12. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{2^{2n+1}(2n)!}$ $= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3}{2}\right)^{2n}}{(2n)!} \left(\frac{3}{2}\right) = \frac{3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \left(\frac{3}{2}\right)^{2n}$

$= \frac{3}{2} \cdot \cos\left(\frac{3}{2}\right)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$T_3(x) \rightsquigarrow x^3$ → $T_n(x)$

13. Find the third degree Taylor polynomial for $f(x) = \sqrt{x}$, centered at $x = 4$. → $a=4$

$\rightsquigarrow x^3$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(4)}{n!}(x-4)^n$$

$$T_3(x) = \frac{f^0(4)}{0!}(x-4)^0 + \frac{f'(4)}{1!}(x-4)^1 + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$n=0$, $n=1$, $n=2$, $n=3 \rightsquigarrow x^3$

$4^{-3/2} = (4^{1/2})^{-3} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

n	$f^n(x)$	$f^n(4)$
0	$\sqrt{x}$	$\sqrt{4} = 2$
1	$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$	$\frac{1}{2\sqrt{4}} = \frac{1}{4}$
2	$(\frac{1}{2})(-\frac{1}{2})x^{-3/2}$	$(-\frac{1}{4})4^{-3/2} = (-\frac{1}{4})(\frac{1}{8}) = -\frac{1}{32}$
3	$(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^{-5/2}$	$\frac{3}{8}\cdot(4^{-5/2}) = (\frac{3}{8})\frac{1}{32} = \frac{3}{256}$

$$T_3(x) = 2 + \frac{1}{4}(x-4) - \left(\frac{1}{64}\right)(x-4)^2 + \left(\frac{1}{512}\right)(x-4)^3$$

14. Find the second degree Taylor polynomial for $f(x) = \arctan(x)$, centered at $x = 1$.

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

n	$f^n(x)$	$f^n(1)$
0	$\arctan(x)$	$\arctan(1) = \pi/4$
1	$\frac{1}{1+x^2}$	$\frac{1}{2}$
2	$\frac{(-1)(2x)}{(1+x^2)^2}$	$\frac{-2}{4} = -\frac{1}{2}$

$$T_2(x) = \frac{\pi}{4} + \left(\frac{1}{2}\right)(x-1) - \frac{1}{4}(x-1)^2$$

→ coefficient on x^2 is $-\frac{1}{4}$

Ratio Test $\quad a=5$

15. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-5)^n}{n!}$

RT: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{3^{n+1}(x-5)^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n \cdot (x-5)^n} \right|$

$= |3(x-5)| \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$

This series will converge for all values of x

Radius : ∞

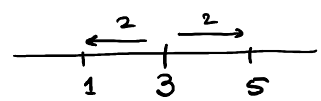
IC : $(-\infty, \infty)$

$(-1)^n(-1)^n = (-1)^{n+n} = (-1)^{2n}$ $\quad a=3$

16. Given that the radius of convergence for the series $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(x-3)^n}{2^n n^4}$ is 2, find the interval of convergence.

Test end points $\quad R=2$

$x=1$ $\quad f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (1-3)^n}{2^n \cdot n^4} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} \cdot 2^n}{2^n \cdot n^4} = \sum_{n=1}^{\infty} \frac{1}{n^4}$



Series converges by p-series.

$x=5$ $\quad f(5) = \sum_{n=1}^{\infty} \frac{(-1)^n (5-3)^n}{2^n \cdot n^4} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n}{2^n \cdot n^4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

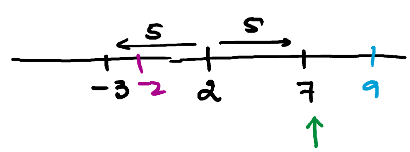
Series converges by Alternating Series Test

IC : $[1, 5]$

17. If the power series $\sum_{n=0}^{\infty} C_n(x-2)^n$ has a radius of convergence of 5. which of the following series will also converge?

$R=5$
$a=2$

(a) $\sum_{n=0}^{\infty} C_n 7^n = \sum_{n=0}^{\infty} C_n(x-2)^n$ $\quad 7 = x-2$, $x=9$ $\}$ Series diverges



(b) $\sum_{n=0}^{\infty} C_n 5^n$ $\quad 5 = x-2$, $x=7$ $\}$ unknown

(c) $\sum_{n=0}^{\infty} (-1)^n C_n 4^n = \sum_{n=0}^{\infty} C_n(-4)^n$ $\}$ series converges.

$-4 = x-2$ or $x=-2$

Tests for convergence/divergence

1. TOD
2. Integral Test
3. CT or LCT
4. AST
5. RT.

18. Which of the following series diverge?

(a) $\sum_{n=2}^{\infty} \frac{n^2-2n-1}{n^3+4n} = \sum a_n$

$\sum b_n = \sum_{n=2}^{\infty} \frac{n^2}{n^3} = \sum_{n=2}^{\infty} \frac{1}{n} \to$ diverges by p-series.

$a_n < b_n$ CT fails

LCT: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^2-2n-1)}{(n^3+4n)} \cdot n = \lim_{n\to\infty} \frac{n^3}{n^3} = 1$

Since $\sum b_n$ diverges, $\sum a_n$ diverges by LCT

(b) $\sum_{n=0}^{\infty} \frac{1}{n^2+2n+4}$ ← $\sum a_n$

$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \to$ converges by p-series

$n^2+2n+4 > n^2$

$\frac{1}{n^2+2n+4} < \frac{1}{n^2}$

$a_n < b_n$ ∴ $\sum a_n$ converges by Comparison Test

(c) $\sum_{n=1}^{\infty} ne^{-n^2} = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$\lim_{n\to\infty} \frac{n}{e^{n^2}} \to \frac{\infty}{\infty}$ $\overset{L'H}{\to} \frac{1}{(2n)e^{n^2}} = \frac{1}{\infty} = 0.$ TOD fails

Int. Test $\int_1^{\infty} \frac{x}{e^{x^2}} dx = \int \frac{1}{e^u} \cdot \frac{du}{2} = \frac{1}{2}\int e^{-u} du = -\frac{1}{2} e^{-u} = -\frac{1}{2} e^{-x^2} \Big|_{x=1}^{x=\infty} = -\frac{1}{2}\left[e^{-\infty}{}^{0} - e^{-1}\right] = \frac{1}{2e}$

$\sum a_n$ converges by Integral Test

(d) $\sum_{n=1}^{\infty} \frac{1}{n!}$

RT: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \frac{1}{\infty} = 0.$

Series converges by Ratio Test.

(e) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \to \frac{1}{n^{3/2}}$ converges by p-series.

19. Which of these series converge absolutely?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum a_n$ → series conditionally convergent

converges by AST.

$\sum b_n = \sum |a_n| = \sum \frac{1}{\sqrt{n}} \to$ diverges by p-series

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{e^{n+1}} = \sum a_n$ → series diverges

$\sum b_n = |a_n| = \sum \frac{e^n}{e^{n+1}} = \sum \frac{1}{e} \to$ diverges by TOD.

$\lim_{n\to\infty} \frac{e^n}{e^{n+1}} \to \frac{e^n}{e^n} = 1. \to$ diverges by AST

(c) $\sum_{n=1}^{\infty} (\sqrt{n+2} - \sqrt{n}) \frac{(\sqrt{n+2}+\sqrt{n})}{(\sqrt{n+2}+\sqrt{n})} = \frac{(n+2)-n}{\sqrt{n+2}+\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n+2}+\sqrt{n}}$ → series diverges

Compare to $b_n = \sum \frac{2}{2\sqrt{n}}$ → diverges.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum a_n$ → series converges conditionally

$\sum b_n = \sum \frac{1}{n} \to$ diverges

converges by AST

(e) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} = \sum a_n$

$\sum b_n = \sum \frac{2^n}{n!}$

RT: $\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n\to\infty} \frac{2}{(n+1)} = 0$

$\sum b_n$ converges

$\sum a_n$ is absolutely convergent.

$n=?$

20. How many terms would be needed to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ to within 2×10^{-9}?

$= 0.000000002$

error $|R_n| < a_{n+1}$

$|R_n| < \frac{1}{(n+1)^5}$

$\frac{1}{(n+1)^5} < 2\times 10^{-9}$

$(n+1)^5 > \frac{1}{2\times 10^{-9}} = 500000000$

$(n+1) > \sqrt[5]{\frac{1}{2\times 10^{-9}}} = 53.92$

$n > 52.92$

or $n = 53$ Ans

21. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ correct to 3 decimal places. (0.001)

solve for n.

$\frac{1}{(n+1)^5} < 0.001$

$(n+1)^5 > 1000$

$4^5 = 1024$

$(3+1)^5 > 1000$

$n = 3$ terms

$S_3 = a_1 + a_2 + a_3$

$S_3 = -1 + \frac{1}{2^5} - \frac{1}{3^5}$

$R_3 = \frac{1}{4^5}$

22. Using the 5^{th} partial sum to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+3)!}$, find the upper bound for the error in the estimate of the sum of the series.

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$

$|R_5| < a_6$

$|a_6| = \frac{6^2}{(6+3)!} = \frac{36}{9!}$ → largest error possible.