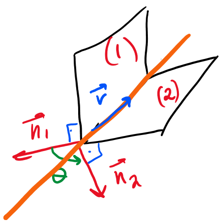
$\vec{n}_1 = \langle 1,-2,1\rangle$ $\quad$ $\vec{n}_2 = \langle 2,1,1\rangle$

1. (a) Find the angle between the planes $x - 2y + z = 1$ and $2x + y + z = 1$.

   (b) Find symmetric equation for the line of intersection of the planes.



(a) angle between planes = angle between $\vec{n}_1$ and $\vec{n}_2$.

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|} = \frac{\langle 1,-2,1\rangle\cdot\langle 2,1,1\rangle}{\sqrt{1+4+1}\cdot\sqrt{4+1+1}} = \frac{2-2+1}{6} = \frac{1}{6}$$

$$\boxed{\theta = \cos^{-1}\left(\tfrac{1}{6}\right) = \arccos\left(\tfrac{1}{6}\right)}$$

(b)

$\vec{v}$ is parallel to the line of intersection

$\vec{v}$ is $\perp$ to both $\vec{n}_1$ and $\vec{n}_2$

$$\Rightarrow \vec{v} = \vec{n}_1\times\vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 1 \\ 1 & 1\end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 1\end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 2 & 1\end{vmatrix}$$

$$\boxed{= -3\vec{i} + \vec{j} + 5\vec{k}}$$ vector parallel to the line.

Point on the line. Find x, y, z such that

$$\begin{cases} x-2y+z=1 \\ 2x+y+z=1 \end{cases}$$ set $z=0$ $$\begin{cases} 2(x-2y)=2(1) \\ 2x+y=1 \end{cases}$$

$$\begin{array}{r} 2x-4y=2 \\ -\quad 2x+y=1 \\ \hline 0-5y=1 \end{array}$$

$\boxed{y=-\frac{1}{5}}$

$x = 1+2y = 1-\frac{2}{5} = \frac{3}{5}$

$\left(\frac{3}{5}, -\frac{1}{5}, 0\right)$ point on the line.

symmetric equations: $$\boxed{\frac{x-3/5}{-3} = \frac{y+1/5}{1} = \frac{z-0}{5}}$$

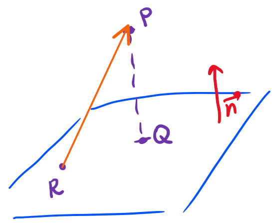
parametric $$\begin{cases} x = \frac{3}{5} - 3t \\ y = -\frac{1}{5} + t \\ z = 5t \end{cases}$$

$\vec{v} = \langle a, b, 0\rangle$ $\qquad$ $\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b},\ z = z_0}$

$$d = \left|\frac{4(1)-6(-2)+2(4)-3}{\sqrt{4^2+6^2+2^2}}\right| = \left|\frac{4+12+8-3}{\sqrt{56}}\right| = \frac{21}{\sqrt{56}}$$

$$4x - 6y + 2z - 3 = 0$$

2. Find the distance from the point $(1, -2, 4)$ to the plane $4x - 6y + 2z = 3$.



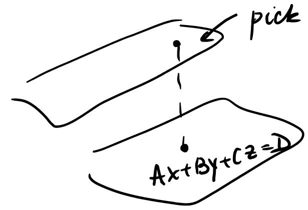
$|PQ| = dist$

$$dist = comp_{\vec{n}} \overrightarrow{RP} = \frac{\overrightarrow{RP} \cdot \vec{n}}{|\vec{n}|}$$

distance from $K(k, l, m)$ to the plane $Ax + By + Cz = D$

$$dist = \left|\frac{Ak + Bl + Cm - D}{\sqrt{A^2+B^2+C^2}}\right|$$

Distance between parallel planes:



pick a point then find the distance to the second plane.

$\frac{x^2}{1}+\frac{y^2}{\frac{1}{4}}+\frac{z^2}{\frac{1}{9}}=1$

3. Match the equation with its graph. Give reasons for your choice.

ellipsoid
a) $x^2+4y^2+9z^2=1$

hyperboloid on one sheet
b) $x^2-y^2+z^2=1$

elliptic paraboloid
c) $y=2x^2+z^2$

d) $x^2+2z^2=1$ (cylinder, elliptic)

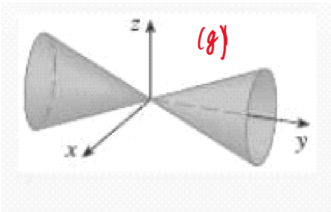
e) $9x^2+4y^2+z^2=1$ (ellipsoid)

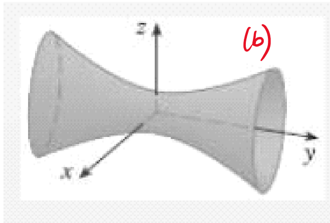
$\frac{x^2}{1/9}+\frac{y^2}{1/4}+\frac{z^2}{1}=1$

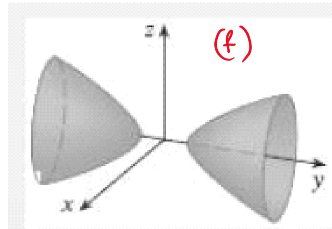
f) $-x^2+y^2-z^2=1$ hyperboloid on two sheets

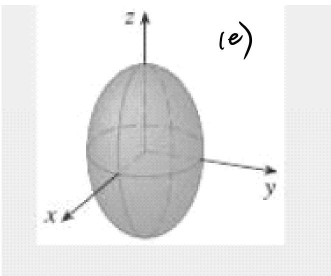
g) $y^2=x^2+2z^2$
cone
elliptic cone

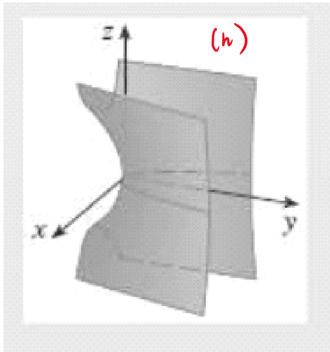
h) $y=x^2-z^2$
hyperbolic paraboloid

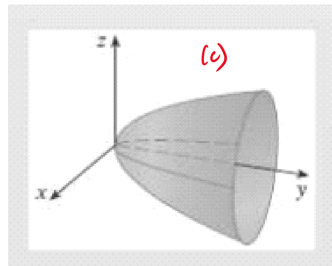


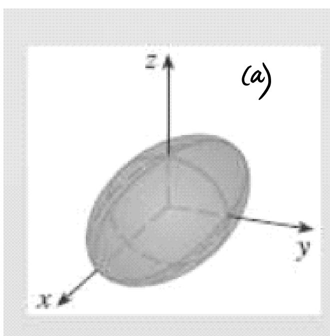


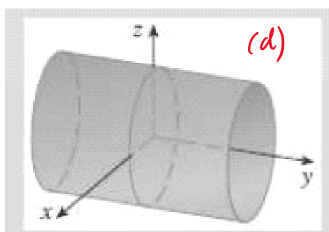












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4. Reduce the equation to the standard form and classify the surface.

(a) $z = (x-1)^2 + (y+5)^2 + 7$ elliptic paraboloid

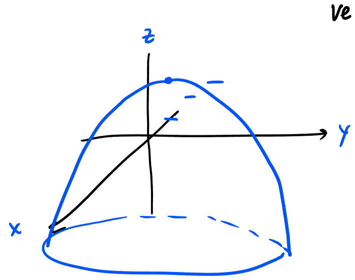
(b) $4x^2 - y^2 + (z-4)^2 = 20$ hyperboloid of one sheet.

(c) $x^2 + y^2 + z + 6x - 2y + 10 = 0$

$(x^2+6x+9) + (y^2-2y+1) + z+10 = 0+9+1$

$(x+3)^2 + (y-1)^2 + z = 0$ or $z = -(x+3)^2 - (y-1)^2$ elliptic paraboloid.

vertex @ $(-3, +1, 0)$, opens down

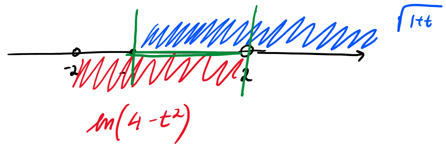


5. Find the domain of $\mathbf{r}(t) =< \ln(4-t^2), \sqrt{1+t}, \sin(\pi t) >$.

$\ln(4-t^2) \longrightarrow 4-t^2>0$ or $t^2<4$ or $\boxed{-2<t<2}$

$\sqrt{1+t} \longrightarrow 1+t \geq 0 \Rightarrow \boxed{t \geq -1}$

$\sin(\pi t) \longrightarrow -\infty < t < \infty$



intersection $[-1, 2) \leftarrow$ domain of $\vec{r}(t)$

6. Find a vector equation for the curve of intersection of the surfaces $x = y^2$ and $z = x$ in terms of the parameter $\boxed{y = t}$.

$x = y^2 = t^2$
$y = t$
$z = x = t^2$

$$\Rightarrow \begin{cases} x = t^2 \\ y = t \\ z = t^2 \end{cases}$$

7. Does the graph of the vector-function $\mathbf{r}(t) = \left\langle \underbrace{\frac{1-t^2}{t}}_{x}, \underbrace{\frac{t+1}{t}}_{y}, \underbrace{t}_{z} \right\rangle$ lie in the plane $x - y + z = -1$?

$$\left[\begin{array}{l} x = \frac{1-t^2}{t} = \frac{1}{t} - t \\ y = \frac{t+1}{t} = 1 + \frac{1}{t} \\ z = t \end{array} \right]$$

plug them into

$x - y + z = -1$

$$\underbrace{\cancel{\frac{1}{t}} - \cancel{t}}_{x} - \underbrace{\left(1 + \cancel{\frac{1}{t}}\right)}_{y} + \underbrace{\cancel{t}}_{z} \overset{?}{=} -1$$

$-1 = -1$

L.H.S. = R.H.S.

$\boxed{\text{yes}}$

8. Find the points where the curve $\mathbf{r}(t) =< \underbrace{1-t}_{x}, \underbrace{t^2}_{y}, \underbrace{t^2}_{z} >$ intersects the plane $5x - y + 2z = -1$.

$$\left. \begin{array}{l} x = 1-t \\ y = t^2 \\ z = t^2 \end{array} \right]$$ plug onto the equation of the plane:

$5x - y + 2z = -1$

$$5\underbrace{(1-t)}_{x} - \underbrace{(t^2)}_{y} + 2\underbrace{t^2}_{z} = -1$$

$$5-5t-t^2+2t^2=-1$$

$$t^2-5t+6=0$$

$$(t-2)(t-3)=0$$

$$t_1=2 \quad , \quad t_2=3$$

Points of intersection are $\vec{r}(2)=\langle 1-2, 2^2, 2^2\rangle = \boxed{(-1,4,4)}$

and $\vec{r}(3)=(1-3, 3^2, 3^2\rangle = \boxed{(-2,9,9)}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

tangent vector $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$.

9. Find parametric equations of the line tangent to the graph of $\mathbf{r}(t) = \langle e^{-t}, t^3, \ln t \rangle$ at the point $t = 1$.

$\vec{r}\,'(t) = \langle -e^{-t}, 3t^2, \frac{1}{t} \rangle$

tangent vector is $\vec{r}\,'(1) = \langle -e^{-1}, 3, 1 \rangle = \langle -\frac{1}{e}, 3, 1 \rangle$ ← vector

point on the curve is $\vec{r}(1) = \langle e^{-1}, 1, \ln 1 \rangle = \langle \frac{1}{e}, 1, 0 \rangle$ ← point.

$$\boxed{\begin{cases} x = \frac{1}{e} - \frac{1}{e}t \\ y = 1 + 3t \\ z = 0 + 1 \cdot t \end{cases}}$$ tangent line.

point, vector

10. Find symmetric equations of the line tangent to the graph of $\mathbf{r}(t) = \left\langle t^2, 4 - t^2, -\dfrac{3}{1+t} \right\rangle$ at the point $(4, 0, 3)$.

$\vec{r}(t) = \langle t^2, 4 - t^2, -\frac{3}{1+t} \rangle$ point $(4, 0, 3)$

tangent vector $\vec{r}\,'(t) = \langle 2t, -2t, \frac{3}{(1+t)^2} \rangle$ ← plug in $t = -2$

Find t such that $\vec{r}(t) = (4, 0, 3)$

$\langle t^2, 4 - t^2, -\frac{3}{1+t} \rangle = (4, 0, 3)$ ← point

$t^2 = 4 \Rightarrow t = \pm 2$

$\vec{r}(2) = \langle 2^2, 4 - 2^2, -\frac{3}{1+2} \rangle = \langle 4, 0, -1 \rangle \Rightarrow t \neq 2$

$\vec{r}(-2) = \langle (-2)^2, 4 - (-2)^2, -\frac{3}{1-2} \rangle = \langle 4, 0, 3 \rangle$

$\boxed{t = -2}$

$\vec{r}\,'(-2) = \langle -4, 4, 3 \rangle$ ← vector

symmetric equations: $\boxed{\dfrac{x-4}{-4} = \dfrac{y-0}{4} = \dfrac{z-3}{3}}$

11. Let

$$\mathbf{r}_1(t) = < \arctan t, t, -t^4 >$$

and

$$\mathbf{r}_2(s) = < s^2 - s, 2\ln s, \frac{\sin(2\pi s)}{2\pi} > .$$

(a) Show that the graphs of the given vector-functions intersect at the origin.

(b) Find their angle of intersection at the origin.

(a) $\vec{r}_1(t) = < \arctan t, t, -t^4 >$

Find t (if possible) such that $\vec{r}_1(t) = (0,0,0)$

$< \arctan t, t, -t^4 > = < 0,0,0 > \Rightarrow \boxed{t=0}$

$\vec{r}_1(0) = (0,0,0)$

Find s (if possible) such that $\vec{r}_2(s) = (0,0,0)$

$\vec{r}_2(s) = < s^2 - s, 2\ln s, \frac{\sin(2\pi s)}{2\pi} > = (0,0,0) \rightarrow \ln s = 0$

$\boxed{s=1}$

$\vec{r}_2(1) = (0,0,0)$

(b) angle of intersection = angle between tangent vectors to $\vec{r}_1$ and $\vec{r}_2$ @ $(0,0,0)$

tangent vectors: $\vec{r}_1{}'(t) = < \frac{1}{1+t^2}, 1, -4t^3 >$, $\vec{r}_1{}'(0) = < 1, 1, 0 >$

$\vec{r}_2{}'(s) = < 2s-1, \frac{2}{s}, \frac{\cos(2\pi s)}{\cancel{2\pi}} \cancel{(2\pi)} >$, $\vec{r}_2{}'(1) = < 1, 2, \cos 2\pi >$

$= < 1, 2, 1 >$

Find an angle between $\vec{r}_1{}'(0)$ and $\vec{r}_2{}'(1)$

$$\cos\theta = \frac{\vec{r}_1{}'(0) \cdot \vec{r}_2{}'(1)}{|\vec{r}_1{}'(0)| \cdot |\vec{r}_2{}'(1)|} = \frac{<1,1,0> \cdot <1,2,1>}{\sqrt{1+1} \cdot \sqrt{1+4+1}} = \frac{1+2}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\boxed{\theta = \frac{\pi}{6}}$$

12. Evaluate the integral $\int_1^4 \left(\sqrt{t}\,\mathbf{i} + te^{-t}\,\mathbf{j} + \frac{1}{t^2}\mathbf{k}\right) dt$

$$= \left\langle \int_1^4 \sqrt{t}\, dt,\ \int_1^4 te^{-t}\, dt,\ \int_1^4 \frac{dt}{t^2} \right\rangle$$

by parts

	D	I
+	t	e^{-t}
−	1	$-e^{-t}$
	0	e^{-t}

$$= \left\langle \frac{t^{3/2}}{3/2}\Big|_1^4,\ \left[-te^{-t} - e^{-t}\right]_1^4,\ -\frac{1}{t}\Big|_1^4 \right\rangle$$

$$= \left\langle \frac{2}{3}\left(4^{3/2}-1\right),\ -4e^{-4} - e^{-4} + e^{-1} + e^{-1},\ -\frac{1}{4} + 1 \right\rangle$$

$$= \left\langle \frac{14}{3},\ -5e^{-4} + 2e^{-1},\ \frac{3}{4} \right\rangle$$

$\vec{r}(t) = \langle \cos^3 t, \sin^3 t, \cos(2t) \rangle$

14. Find the length of the curve given by the vector function $\mathbf{r}(t) = \cos^3 t\, \mathbf{i} + \sin^3 t\, \mathbf{j} + \cos(2t)\, \mathbf{k}$, $\boxed{0 \le t \le \frac{\pi}{2}}$.

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,'(t) = \langle 3\cos^2 t(-\sin t),\ 3\sin^2 t(\cos t),\ -2\sin(2t) \rangle$$

$$= \langle -3\cos^2 t \sin t,\ 3\sin^2 t \cos t,\ -2\sin 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 4\sin^2 2t}$$

$$= \sqrt{9\cos^2 t \sin^2 t(\sin^2 t + \cos^2 t) + 4\sin^2 2t}$$

$$= \sqrt{9\cos^2 t \sin^2 t + 4\sin^2 2t}$$

$$(\sin 2t)^2 = (2\sin t \cos t)^2 = 4\sin^2 t \cos^2 t$$

$$= \sqrt{9\cos^2 t \sin^2 t + 16\sin^2 t \cos^2 t}$$

$$= \sqrt{25\cos^2 t \sin^2 t} = 5\cos t \sin t$$

$$L = \int_0^{\pi/2} 5\cos t \sin t\,dt$$

$$\left|\begin{array}{l} u = \sin t \\ du = \cos t\,dt \\ t = \frac{\pi}{2} \to u = \sin\frac{\pi}{2} = 1 \\ t = 0 \to u = \sin 0 = 0 \end{array}\right|$$

$$= \int_0^1 5u\,du = \left.\frac{5u^2}{2}\right|_0^1 = \boxed{\frac{5}{2}}$$

15. For the curve given by $\mathbf{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t \rangle$, $0 \le t \le \dfrac{\pi}{2}$, find

(a) the unit tangent vector $\mathbf{T}(t)$

(b) the unit normal vector $\mathbf{N}(t)$

~~(c) the binormal vector $\mathbf{B}(t)$~~

(d) the curvature

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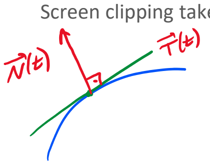

tangent vector $\quad \vec{r}'(t) = \langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t \rangle$

unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$|\vec{r}'(t)| = \sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t + 4\sin^2 t\cos^2 t}$$

$$= \sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t) + 4\sin^2 t\cos^2 t}$$

$$= \sqrt{13\sin^2 t\cos^2 t} = \boxed{\sqrt{13}\sin t\cos t}$$

$|\vec{r}'(t)|$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t \rangle}{\sqrt{13}\sin t\cos t} = \left\langle \frac{3}{\sqrt{13}}\sin t, -\frac{3}{\sqrt{13}}\cos t, \frac{2}{\sqrt{13}} \right\rangle$$

unit normal vector $\quad \vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$

$$\vec{T}'(t) = \left\langle \frac{3}{\sqrt{13}}\cos t, +\frac{3}{\sqrt{13}}\sin t, 0 \right\rangle$$

$$|\vec{T}'(t)| = \sqrt{\frac{9}{13}\cos^2 t + \frac{9}{13}\sin^2 t} = \sqrt{\frac{9}{13}(\cos^2 t + \sin^2 t)} = \sqrt{\frac{9}{13}} = \boxed{\frac{3}{\sqrt{13}}} \leftarrow |\vec{T}'(t)|$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\left\langle \frac{3}{\sqrt{13}}\cos t, \frac{3}{\sqrt{13}}\sin t, 0 \right\rangle}{\frac{3}{\sqrt{13}}} = \langle \cos t, \sin t, 0 \rangle$$

The curvature $\quad \kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{3}{\sqrt{13}}}{\sqrt{13}\sin t\cos t} = \boxed{\frac{3}{13\sin t\cos t}}$$