Math 251 Fall 2022 Chapter 16

Disclaimer: Math 251 is a coordinated class, therefore the rules are the same across ALL sections/professors. No formulas will be given on the final exam, and you cannot use a formula sheet. No calculators are allowed.

Use this as a guide to help you organize your thoughts on Chapter 16. You also should review double integrals, triple integrals, and all techniques of integration used in these sections, both in lecture and homework. Specifically, u substitution and trigonometric integrals of the form $\int \cos^n x \sin^m x\, dx$.

As you begin to prepare for your final exam, hand write a few examples done in lecture over EACH CONCEPT numbered below. Next, completely randomize every problem done in lecture from chapter 16. Old school style cut with scissors and draw from a hat, not so old school, screencapture each question and randomize that. Then choose a problem AT RANDOM. Lastly, once you feel prepared, take the 'Former Final Exam' I posted. Work it in a *timed* 2 hour time, and at the end of the 2 hours, use the posted solutions to see what you need to work on.

1. If f is defined on a smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, then the line integral of f along C is

$$\int_C f(x,y)ds = \int_a^b (f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b (f(x(t), y(t))|\mathbf{r}'(t)|\, dt.$$

2. If f is defined on a smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$, then the line integral of f along C is

$$\int_C f(x,y,z)ds = \int_a^b (f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b (f(x(t), y(t), z(t))|\mathbf{r}'(t)|\, dt.$$

3. Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$.

   a.) The line integral of f along C with respect to x is $\int_C f(x,y)dx = \int_a^b (f(x(t), y(t))\ x'(t)\ dt.$

   b.) The line integral of f along C with respect to y is $\int_C f(x,y)dy = \int_a^b (f(x(t), y(t))\ y'(t)\ dt.$

4. Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$.

   The line integral of f along C with respect to z is $\int_C f(x,y,z)dz = \int_a^b (f(x(t), y(t), z(t))\ z'(t)\ dt.$

5. The line integral of $\mathbf{F}$ along a curve C parameterized by $\mathbf{r}(t)$, $a \le t \le b$, is $\int_c \mathbf{F}\cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt.$

6. Test for conservative vector fields

   a.) $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = P\mathbf{i}+Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

   b.) If $\mathbf{F}$ is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

7. Fundamental Theorem for Line Integrals: Let C be a smooth curve parameterized by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C.

   Then $\displaystyle\int_c \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

8. Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then $\displaystyle\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$.

9. If $\mathbf{F} = \langle P, Q, R\rangle$ is a a vector field on $\Re^3$ and the partial derivatives of P, Q, and R all exist, the **del operator**, denoted by ∇, is $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$.

10. The Divergence of F is $\nabla \cdot \mathbf{F} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle \cdot \langle P, Q, R\rangle$.

11. The Curl of $\mathbf{F}$ is the vector field on $\Re^3$ defined by curl $\mathbf{F} = \nabla \times \mathbf{F}$.

12. If a smooth parametric surface S is parameterized by $\mathbf{r}(u,v)$, and S is covered just once as (u,v) ranges throughout the parametric domain D, then the **surface area** of S is $A(S) = \displaystyle\iint\limits_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$.

13. If a smooth parametric surface S is parameterized by $\mathbf{r}(u,v)$, and S is covered just once as (u,v) ranges throughout the parametric domain D, then the surface integral of f over S is given by $\displaystyle\iint_S f(x,y,z)\, dS = \iint\limits_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$.

14. Let $\mathbf{F}$ be a vector field whose domain includes the positively oriented surface S, where S is defined parametrically by $\mathbf{r}(u,v)$, $u, v \in D$. Then the surface integral of $\mathbf{F}$ over S, also called the **Flux** of $\mathbf{F}$ over S, is $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$.

15. **Stokes' Theorem:** Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive (counterclockwise) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains S. Then $\displaystyle\int\limits_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$

16. **Divergence Theorem** Let E be a simple solid region whose boundary surface has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then $\displaystyle\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_E \text{div } \mathbf{F}\, dV$