1. Given the sequence $\{a_n\} = \left\{ \overset{a_1}{2}, \overset{a_2}{\frac{3}{4}}, \overset{a_3}{\frac{4}{9}}, \overset{a_4}{\frac{5}{16}}, \cdots \right\}$

(a) Find the formula for the n-th term a_n

$1 = 1^2 \leftarrow a_1 = 2 \leftarrow n=1 \qquad 2 = 1+1$

$4 = 2^2 \leftarrow a_2 = \frac{3}{4} \leftarrow n=2 \qquad 3 = 2+1$

$9 = 3^2 \leftarrow a_3 = \frac{4}{9} \leftarrow n=3 \qquad 4 = 3+1$

$16 = 4^2 \leftarrow a_4 = \frac{5}{16} \leftarrow n=4 \qquad 5 = 4+1$

$\boxed{a_n = \frac{n+1}{n^2}}$

(b) Find the limit of the sequence.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{\cancel{n}\cdot\frac{n+1}{n}}{n^{\cancel{2}}} = \lim_{n\to\infty} \frac{\frac{n}{n}+\frac{1}{n}}{n} = \lim_{n\to\infty} \frac{1+\overset{0}{\cancel{\frac{1}{n}}}}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\boxed{\lim_{n\to\infty} a_n = 0}$ (convergent).

(c) Is this a monotonic sequence?

$2 > \frac{3}{4} > \frac{4}{9} > \frac{5}{16} > \cdots$

for all $n \geq 1$ $\quad a_{n+1} < a_n$ — $\boxed{\text{monotonic}}$

max term of the sequence.

(d) Is this a bounded sequence?

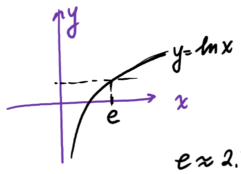
$\boxed{0 < a_n < 2}$ hence $\boxed{\text{bounded.}}$

since $a_n = \frac{n+1}{n^2}$, and n is a positive integer, then $a_n > 0$.

$n > 0,\ n = 1, 2, 3, \ldots$

2. Is the sequence given by $a_n = \dfrac{\ln n}{n}$ increasing or decreasing? Find the limit of the sequence.

$$a_n' = \left(\frac{\ln n}{n}\right)' = \frac{\frac{1}{n}\cdot n - \ln n}{n^2} = \frac{1 - \ln n}{n^2} < 0, \text{ if } n \geq 3$$



$e \approx 2.7$

$\boxed{\text{decreasing}}$ for $n \geq 3$.

$$a_1 = \frac{\cancel{\ln 1}^{\,0}}{1} = 0$$

$$a_2 = \frac{\ln 2}{2} \approx 0.346573$$

$$\ldots < a_4 < a_3 < a_2$$

3. Find the limit of the sequence

(a) $a_n = \cos(\pi n)$

$\cos 0 = 1$

$\cos \pi = -1$

$\cos 2\pi = 1$

$\cos 3\pi = -1$

$\ldots$

$\cos \pi n = (-1)^n$

n is even $\rightarrow a_n = 1$

n is odd $\rightarrow a_n = -1$

$1, -1, 1, -1, \ldots$ not monotonic.

$\boxed{\text{divergent}}$

$\lim_{n\to\infty} \cos \pi n$ DNE.

(b) $a_n = \dfrac{5-2n-3n^2}{6n^2+n-6}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5-2n\boxed{-3n^2}}{\boxed{6n^2}+n-6} = \lim_{n\to\infty} \frac{\cancel{n^2}\,\frac{5-2n-3n^2}{n^2}}{\cancel{n^2}\,\frac{6n^2+n-6}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\cancel{\frac{5}{n^2}}^{0} - \frac{2\cancel{n}}{n^{\cancel{2}}} - \frac{3\cancel{n^2}}{\cancel{n^2}}}{\frac{6\cancel{n^2}}{\cancel{n^2}} + \cancel{\frac{n}{n^2}}^{0} - \cancel{\frac{6}{n^2}}^{0}} = -\frac{3}{6} = -\frac{1}{2}.$$

(c) $\ln a_n = \ln\left(1+\frac{3}{n}\right)^{2n}$

$$\ln a_n = 2n \ln\left(1+\frac{3}{n}\right)$$

$$\lim_{n\to\infty} 2n^{\infty} \ln\left(1+\cancel{\frac{3}{n}}^{0}\right) = |0\cdot\infty| = 2\lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{n}} \quad \left|\frac{0}{0}\right|$$

L'Hospital's Rule $\quad 2\lim_{n\to\infty} \dfrac{\frac{1}{1+\frac{3}{n}}\left(+\frac{3}{\cancel{n^2}}\right)}{+\frac{1}{\cancel{n^2}}} = 2\lim_{n\to\infty} \dfrac{3}{1+\cancel{\frac{3}{n}}^{0}} = 6$

$\lim_{n\to\infty}(\ln a_n) = 6 \quad$ hence $\quad \boxed{\lim_{n\to\infty} a_n = e^6}$

(d) $a_n = \dfrac{4+(-1)^n}{n}$

if n is even, then $a_n = \dfrac{4+(-1)^n}{n} = \dfrac{4+1}{n} = \dfrac{5}{n}$

$$\lim_{n\to\infty} \frac{5}{n} = 0$$

if n is odd, then $(-1)^n = -1$ and $a_n = \dfrac{4-1}{n} = \dfrac{3}{n}$

$$\lim_{n\to\infty} \frac{3}{n} = 0$$

Both limits match, thus $\boxed{\lim_{n\to\infty} \frac{4+(-1)^n}{n} = 0}$

4. Find the limit of the sequence $\{a_n\}$ given recursively

$$a_1 = 2, \qquad a_{n+1} = 1 - \frac{1}{a_n}$$

if it converges.

$a_1 > a_2 > a_3 < a_4$

$a_1 = 2$

$a_2 = a_{1+1} = 1 - \frac{1}{a_1} = 1 - \frac{1}{2} = \frac{1}{2}$

$a_3 = a_{2+1} \overset{n=2}{=\!=} 1 - \frac{1}{a_2} = 1 - \frac{1}{1/2} = 1 - 2 = -1$

$a_4 = a_{3+1} \overset{n=3}{=\!=} 1 - \frac{1}{a_3} = 1 - \frac{1}{-1} = 2$

$a_5 = \frac{1}{2}$

$a_6 = -1$

$a_7 = 2$

not monotonic, not convergent.

Denote $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(1 - \frac{1}{a_n}\right)$

$L = 1 - \frac{1}{L}$ or $L^2 = L - 1$ or $L^2 - L + 1 = 0$. solve for L.

5. Find the limit of the sequence $\boxed{a_n = \dfrac{(-1)^n(n^2+2)}{3n^2+1}}$ or show that it diverges.

n is even, $(-1)^n = 1$, and $a_n = \frac{n^2+2}{3n^2+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\boxed{n^2}+2}{\boxed{3n^2}+1} = \frac{1}{3}$

do not match.

n is odd, $(-1)^n = -1$, and $a_n = \frac{-(n^2+2)}{3n^2+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-(n^2+2)}{3n^2+1} = -\frac{1}{3}$

the limits don't match $\Longrightarrow$ the sequence is divergent.

$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + \dots + a_n}_{\sum_{k=1}^{n} a_n = S_n} + \dots$ a partial sum.

6. Find the third partial sum s_3 of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$. $= \underbrace{\cos\pi}_{n=1} + \underbrace{\frac{\cos 2\pi}{2}}_{n=2} + \underbrace{\frac{\cos 3\pi}{3}}_{n_3} + \dots$

$S_3 = a_1 + a_2 + a_3 = \overset{-1}{\cancel{\cos\pi}} + \overset{1}{\frac{\cancel{\cos 2\pi}}{2}} + \overset{-1}{\frac{\cancel{\cos 3\pi}}{3}} = -1 + \frac{1}{2} - \frac{1}{3} = \frac{1}{2} - \frac{4}{3} = \frac{3-8}{6} = \boxed{-\frac{5}{6}}$

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_{n-1} + a_n + \dots$

$S_n = \underbrace{a_1 + a_2 + \dots + a_{n-1}}_{S_{n-1}} + a_n \longrightarrow S_n = S_{n-1} + a_n \longrightarrow a_n = S_n - S_{n-1}$

$S_{n-1} = a_1 + a_2 + \dots + a_{n-1}$

7. The n-th partial sum of the series is $s_n = 3 - n2^{-n}$. $= 3 - \frac{n}{2^n}$

(a) Find a_n

$a_n = S_n - S_{n-1} = \underbrace{\cancel{3} - \frac{n}{2^n}}_{S_n} - \underbrace{\left(\cancel{3} - \frac{n-1}{2^{n-1}}\right)}_{S_{n-1}} = -\frac{n}{2^n} + \frac{n-1}{2^n \cdot 2^{-1}} = -\frac{n}{2^n} + \frac{2(n-1)}{2^n}$

$= \frac{2n-2-n}{2^n} = \boxed{\frac{n-2}{2^n} = a_n}$

(b) Find the sum of the series.

S_n is the nth partial sum of the series then the sum of the series $\boxed{S = \lim_{n\to\infty} S_n}$

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(3 - \frac{n}{2^n}\right) = 3 - \lim_{n\to\infty} \frac{n}{2^n} \left(\frac{\infty}{\infty}\right)$

L'Hospital's Rule

$3 - \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 3$ (limit $\to 0$)

$\boxed{S = 3}$

8. Find the n-th partial sum of the series $\sum_{n=1}^{\infty} \frac{3}{n^2+n}$. Does the series converge? If yes, find the sum of the series?

Step 1. Partial fractions $\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$$\frac{1}{n(n+1)} = \frac{A(n+1)+Bn}{n(n+1)}$$

$$1 = A(n+1) + Bn$$

$n=0$ $\boxed{1 = A}$

$n=-1$ $1 = -B \rightarrow \boxed{B=-1}$

$$\frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{3}{n^2+n} = 3\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Step 2. Find a formula for the n-th partial sum S_n.

$$a_1 = S_1 = 3\left(\frac{1}{1} - \frac{1}{1+1}\right) = 3\left(1 - \frac{1}{\underset{1+1}{2}}\right)$$

$$a_1 + a_2 = S_2 = 3\left(\underbrace{1 - \cancel{\frac{1}{2}}}_{n=1} + \underbrace{\cancel{\frac{1}{2}} - \frac{1}{3}}_{n=2}\right) = 3\left(1 - \frac{1}{\underset{2+1}{3}}\right)$$

$$S_2 + a_3 = a_1 + a_2 + a_3 = S_3 = \underbrace{3\left(1 - \frac{1}{3}\right)}_{S_2} + \underbrace{3\left(\frac{1}{3} - \frac{1}{4}\right)}_{a_3} = 3\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4}\right) = 3\left(1 - \frac{1}{\underset{3+1}{4}}\right)$$

$$\boxed{S_n = 3\left(1 - \frac{1}{n+1}\right)}$$

Step 3. $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 3\left(1 - \frac{1}{n+1}\right) = \boxed{3}$

$\boxed{\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}}$, if $|r|<1$

9. Find the sum of the series $\sum_{n=3}^{\infty} 10\left(\frac{2}{5}\right)^{n-1}$. $= \underbrace{10\left(\frac{2}{5}\right)^{3-1}}_{n=3} + \underbrace{10\left(\frac{2}{5}\right)^{4-1}}_{n=4} + \underbrace{10\left(\frac{2}{5}\right)^{5-1}}_{n=5} + \dots$

$= 10\left(\frac{2}{5}\right)^2 + 10\left(\frac{2}{5}\right)^3 + 10\left(\frac{2}{5}\right)^4 + \dots$

$= 10\left(\frac{2}{5}\right)^2\left[1\left(\frac{2}{5}\right)^0 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \dots\right]$

$= 10 \cdot \frac{4}{25} \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{8}{5} \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

$\overset{r=2/5}{\underset{a=1}{=}} \frac{8}{5} \cdot \frac{1}{1-\frac{2}{5}} = \frac{8}{5} \cdot \frac{1}{\frac{3}{5}}$

$= \frac{8}{5} \cdot \frac{5}{3} = \boxed{\frac{8}{3}}$

both series converge

10. Find the sum of the series $\sum_{n=0}^{\infty} \left[\frac{1}{5^n} + \left(\frac{2}{3}\right)^n\right]$.

geometric $r = \frac{1}{5} < 1$

geometric $r = \frac{2}{3} < 1$

$= \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

$= \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{4}{5}} + \frac{1}{\frac{1}{3}}$

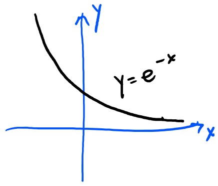
$= \frac{5}{4} + 3 = \boxed{\frac{17}{4}}$

Test for divergence. For a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

11. Determine whether the series is convergent or divergent. If it converges, find its sum.

(a) $\sum_{n=0}^{\infty} \frac{1}{4+e^{-n}}$

$\lim_{n\to\infty} \frac{1}{4+\cancel{e^{-n}}\to 0} = \frac{1}{4} \neq 0$



The series is divergent by the divergence test.

(b) $\sum_{n=0}^{\infty} \frac{2^n+1}{e^n} = \sum_{n=0}^{\infty}\left(\frac{2^n}{e^n}+\frac{1}{e^n}\right) = \sum_{n=0}^{\infty}\left[\left(\frac{2}{e}\right)^n + \left(\frac{1}{e}\right)^n\right]$

$r=\frac{2}{e}<1$ $r=\frac{1}{e}<1$

geometric series.

convergent.

$= \sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n = \frac{1}{1-\frac{2}{e}} + \frac{1}{1-\frac{1}{e}} = \frac{e}{e-2} + \frac{e}{e-1}$

(c) $\sum_{n=0}^{\infty} \frac{e^n}{n^3}$ $\boxed{\text{divergent}}$

$\lim_{n\to\infty} \frac{e^n}{n^3} \left(\frac{\infty}{\infty}\right) \overset{\text{L'Hospital's Rule}}{=} \lim_{n\to\infty} \frac{e^n}{3n^2} \overset{\text{L'Hospital's Rule}}{=} \lim_{n\to\infty} \frac{e^n}{6n}$

$\overset{\text{L'Hospital's Rule}}{=} \lim_{n\to\infty} \frac{e^n}{6} = \infty$

$\boxed{\text{divergent by the test of divergence.}}$