Math 151
Week-In-Review 7
K.1, K.2, 3.7, [redacted]

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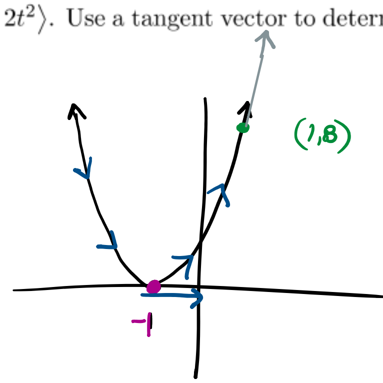
Problem Statements

1. Sketch a curve with the vector equation $\boldsymbol{r}(t) = \langle t-1, 2t^2 \rangle$. Use a tangent vector to determine the direction of the curve as t increases.

$x = t - 1 \Rightarrow t = x+1$

$y = 2t^2$

$y = 2(x+1)^2$



$\vec{r}'(t) = \langle 1, 4t \rangle$

$\vec{r}(0) = \langle 0-1, 2(0)^2 \rangle = \langle -1, 0 \rangle$

$\vec{r}'(0) = \langle 1, 4(0) \rangle = \langle 1, 0 \rangle$

$\vec{r}'(2) = \langle 1, 4(2) \rangle = \langle 1, 8 \rangle$

$\vec{r}(2) = \langle 2-1, 2(2)^2 \rangle = \langle 1, 8 \rangle$

2. If $\boldsymbol{r}(t) = \langle t\cos(t), 4\sin(t)\rangle$ represents the position of a particle at time t, find the velocity, acceleration, and speed of the particle when $t = \frac{\pi}{3}$.

Position : $\vec{r}(t) = \langle t\cos(t), 4\sin(t)\rangle$

Velocity : $\vec{v}(t) = \vec{r}\,'(t) = \langle t\cdot(-\sin t) + \cos(t)\cdot 1, \; 4\cos(t)\rangle$

Acceleration: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \langle t\cdot(-\cos t) + (-\sin t)\cdot 1 - \sin(t), \; -4\sin(t)\rangle$

$= \langle -t\cos(t) - 2\sin(t), \; -4\sin(t)\rangle$

$\vec{v}\left(\frac{\pi}{3}\right) = \left\langle \frac{\pi}{3}\cdot\left(-\sin\left(\frac{\pi}{3}\right)\right) + \cos\left(\frac{\pi}{3}\right), \; 4\cos\left(\frac{\pi}{3}\right)\right\rangle = \left\langle \frac{\pi}{3}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2}, \; 4\left(\frac{1}{2}\right)\right\rangle$

$$\boxed{\vec{v}\left(\frac{\pi}{3}\right) = \left\langle \frac{-\pi\sqrt{3}}{6} + \frac{1}{2}, \; 2\right\rangle}$$

$\vec{a}\left(\frac{\pi}{3}\right) = \left\langle -\frac{\pi}{3}\cos\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{\pi}{3}\right), -4\sin\left(\frac{\pi}{3}\right)\right\rangle = \left\langle -\frac{\pi}{3}\left(\frac{1}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right), -4\left(\frac{\sqrt{3}}{2}\right)\right\rangle$

$$\boxed{\vec{a}\left(\frac{\pi}{3}\right) = \left\langle -\frac{\pi}{6} - \sqrt{3}, \; -2\sqrt{3}\right\rangle}$$

Speed : $|\vec{v}(t)|$

$$\boxed{\left|\vec{v}\left(\frac{\pi}{3}\right)\right| = \sqrt{\left(\frac{-\pi\sqrt{3}}{6} + \frac{1}{2}\right)^2 + (2)^2}}$$

3. Find a unit tangent vector to the curve $\boldsymbol{r}(t) = \langle 2te^{5t}, t^t \rangle$ when $t = 2$.

$$\vec{r}\,'(t) = \langle 2t \cdot e^{5t} \cdot 5 + e^{5t} \cdot 2,\ [1 + \ln(t)]\, t^t \rangle$$

$y = t^t$ Find $\frac{dy}{dt}$

Logarithmic Differentiation

(Uses Implicit Differentiation)

$$\ln(y) = \ln(t^t)$$

$$\ln(y) = t \cdot \ln(t)$$

$$\frac{d}{dt}[\ln(y)] = \frac{d}{dt}[t \cdot \ln(t)]$$

$$\frac{1}{y} \cdot \frac{dy}{dt} = t \cdot \frac{1}{t} + \ln(t) \cdot 1$$

$$\frac{dy}{dt} = [1 + \ln(t)] \cdot y$$

$$\frac{dy}{dt} = [1 + \ln(t)]\, t^t$$

Tangent Vector when $t = 2$

$$\vec{r}\,'(2) = \langle 2(2)e^{5(2)} \cdot 5 + e^{5(2)} \cdot 2,\ [1 + \ln(2)]\, 2^2 \rangle$$

$$= \langle 20e^{10} + 2e^{10},\ [1 + \ln(2)] \cdot 4 \rangle$$

$$= \langle 22e^{10},\ 4(1 + \ln(2)) \rangle$$

Unit Tangent Vector when $t = 2$

$$\boxed{\frac{\vec{r}\,'(2)}{|\vec{r}\,'(2)|} = \frac{\langle 22e^{10},\ 4(1 + \ln(2)) \rangle}{\sqrt{(22e^{10})^2 + [4(1 + \ln(2))]^2}}}$$

$\vec{r}(t) = \langle t^3 - 5t, t^2 \rangle$

4. A curve has parametric equations $x = t^3 - 5t, y = t^2$.

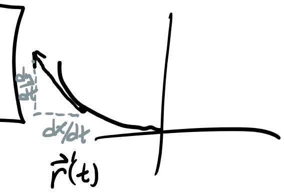
Slope $= \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

(a) Find an expression for the slope of the tangent line to the curve, where it is defined.

$\frac{dx}{dt} = 3t^2 - 5$

$\frac{dy}{dt} = 2t$

Slope: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{2t}{3t^2 - 5}}$



(b) Find Cartesian equations of *both* tangent lines to the curve that pass through the point $(0, 5)$.

First find both t-values that correspond with $(0,5)$

$t^3 - 5t = 0$

$t(t^2 - 5) = 0$

$\cancel{t=0}$ $\quad t^2 = 5$

$t = \pm\sqrt{5}$

$t^2 = 5$

$t = \pm\sqrt{5}$

Point: $(0, 5)$

Slope: $m = \frac{dy}{dx} = \frac{2t}{3t^2 - 5}$

$t = \sqrt{5}$: $\frac{dy}{dx} = \frac{2\sqrt{5}}{3(\sqrt{5})^2 - 5} = \frac{2\sqrt{5}}{10} = \frac{\sqrt{5}}{5}$

$t = -\sqrt{5}$: $\frac{dy}{dt} = \frac{2(-\sqrt{5})}{3(-\sqrt{5})^2 - 5} = \frac{-2\sqrt{5}}{10} = \frac{-\sqrt{5}}{5}$

$$\boxed{\begin{aligned} y - 5 &= \frac{\sqrt{5}}{5}(x - 0) \\ y - 5 &= \frac{-\sqrt{5}}{5}(x - 0) \end{aligned}}$$

(c) Find the points where the tangent line to the curve is horizontal or vertical.

$\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$

Horizontal Tangent: $\frac{dy}{dt} = 0$

$x = t^3 - 5t$
$y = t^2$

Horizontal:

$\frac{dy}{dt} = 2t = 0$

$t = 0$

$\boxed{(0, 0)}$

Vertical Tangent: $\frac{dx}{dt} = 0$

$3t^2 - 5 = 0$

$3t^2 = 5$

$t^2 = 5/3$

$t = \pm\sqrt{5/3}$

$$\boxed{\begin{aligned} &\left(\left(\sqrt{5/3}\right)^3 - 5\sqrt{5/3},\ 5/3\right) \\ &\left(\left(-\sqrt{5/3}\right)^3 - 5\left(-\sqrt{5/3}\right),\ 5/3\right) \end{aligned}}$$

5. The graph of the parametric equations $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$ is called a cycloid.

   (a) What are the coordinates of x and y when $\theta = 0$?

$x = r(0 - \sin(0)) = r \cdot 0 = 0$ $\qquad (0,0)$

$y = r(1 - \cos(0)) = r(1-1) = 0$

   (b) Show that this cycloid has a vertical tangent at the origin.

$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \frac{dy}{d\theta} = r(0 - (-\sin\theta)) = r\sin\theta$

$\theta = 0$: $\frac{dx}{d\theta} = r(1 - \cos(0)) = r(1-1) = 0 \qquad \frac{dy}{d\theta} = r\sin(0) = 0$

?

Vertical Tangent or Horizontal Tangent?

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad \lim_{\theta\to 0} \frac{dy/d\theta}{dx/d\theta} = \lim_{\theta \to 0} \frac{\cancel{r}\sin\theta}{\cancel{r}(1-\cos\theta)} \cdot \frac{(1+\cos\theta)}{(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta\,(1+\cos\theta)}{1-\cos^2\theta} = \lim_{\theta\to 0}\frac{\sin\theta\,(1+\cos\theta)}{\sin^2\theta}$$

$$= \lim_{\theta\to 0}\frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos(0)}{\sin(0)} = \frac{2}{0}$$

Δy

Δx

$\frac{\#}{0}$ $\qquad \infty$

Note: We didn't really finish proving this. We just discussed this approaching an infinite slope. You shouldn't see it on an exam.
But also I don't write the exams.

6. The position of a particle is represented by $f(t) = \frac{9t}{t^2+9}$, $t \geq 0$, where t is measured in seconds and $f(t)$ is measured in feet.

(a) Find the velocity at time t.

$$v(t) = f'(t) = \frac{(t^2+9)(9)-(9t)(2t)}{(t^2+9)^2} = \frac{9t^2+81-18t^2}{(t^2+9)^2} = \frac{81-9t^2}{(t^2+9)^2} \text{ ft/s}$$

(b) What is the velocity after 1 second?

$$v(1) = \frac{81-9(1)^2}{(1^2+9)^2} = \frac{72}{10^2} = \boxed{\frac{72}{100}} \text{ ft/s}$$

(c) When is the particle at rest? Velocity $= 0$

$$\frac{81-9t^2}{(t^2+9)^2} = 0$$

$$81-9t^2 = 0$$

$$81 = 9t^2$$

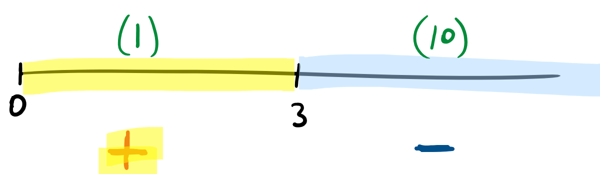
$$9 = t^2$$

$$t = \pm 3$$

$$\boxed{t = 3}$$

(d) When is the particle moving in the positive direction? $v(t) > 0$

$$\frac{81-9t^2}{(t^2+9)^2} > 0$$



$$v(1) = \frac{72}{100}$$

$$v(10) = \frac{81-9(10)^2}{(10^2+9)^2} = \frac{-}{+}$$

$v(t)$:

Positive Direction : $[0,3)$

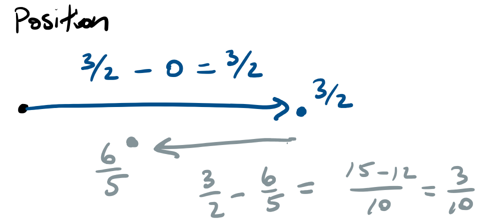
Negative Direction : $(3,\infty)$

(e) Find the total distance traveled in the first 6 seconds.

$f(t) = \dfrac{9t}{t^2+9}$

$v(t) = \dfrac{81-9t^2}{(t^2+9)^2}$

t	$f(t)$
0	0
3	$27/18 = 3/2$
6	$54/45 = 6/5$



$$\boxed{\text{Distance: } \frac{3}{2} + \frac{3}{10}}$$

(f) Find the acceleration at time t.

$$a(t) = v'(t) = \frac{(t^2+9)^2\cdot(-18t) - (81-9t^2)\cdot 2(t^2+9)\cdot(2t)}{(t^2+9)^4} = \frac{(t^2+9)\left[(t^2+9)(-18t) - (81-9t^2)4t\right]}{(t^2+9)^4}$$

$$= \frac{-18t^3 - 162t - 324t + 36t^3}{(t^2+9)^3} = \frac{18t^3 - 486t}{(t^2+9)^3} = \frac{18t(t^2-27)}{(t^2+9)^3}$$

(g) When is the particle accelerating in the positive direction?

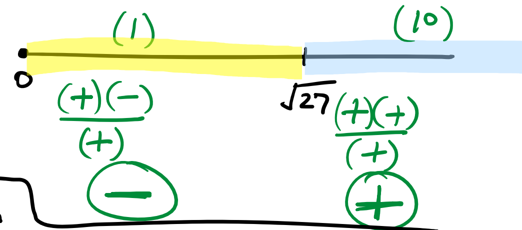
$a(t) > 0$

$a(t) = 0$

$$\frac{18t(t^2-27)}{(t^2+9)^3} = 0$$

$$18t(t^2-27) = 0$$

$t = 0 \qquad t^2 = 27 \qquad t = \pm\sqrt{27}$



$$\boxed{\text{Acceleration in Positive Direction: } (\sqrt{27}, \infty)}$$

(h) When is the particle speeding up?

Speeding Up

$\cancel{v(t) > 0 \text{ and } a(t) > 0}$ OR $\left.\begin{array}{c} v(t) < 0 \\ \text{and} \\ a(t) < 0 \end{array}\right\} \Rightarrow \boxed{(3, \sqrt{27})}$

(i) When is the particle slowing down?

$$\text{Slowing: } \boxed{[0, 3) \cup (\sqrt{27}, \infty)}$$

7. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, the Torricelli's Law gives the volume, V, of water remaining in the tank after t minutes according to the formula: $V = 5000\left(1 - \frac{1}{40}t\right)^2$, $0 \leq t \leq 40$.

   (a) Find the rate at which water is draining from the tank after t minutes.

$$\frac{dV}{dt} = 5000 \cdot 2\left(1-\frac{1}{40}t\right)\left(-\frac{1}{40}\right)$$

$$= \frac{10000\left(1-\frac{1}{40}t\right)}{-40} = \boxed{-250\left(1-\frac{1}{40}t\right)}$$

V'

   (b) Find the rate at which the water is draining after 5, 10, 20, and 40 minutes.

$V'(5) = -218.75$

$V'(10) = -187.5$

$V'(20) = -125$

$V'(40) = 0$

   (c) At what time is water flowing out the fastest? The slowest?

$t=0$ $\qquad$ $t=40$

8. A ball is thrown vertically upward with a velocity of 80ft/s. Its height after t seconds is $s = 80t - 16t^2$.

   (a) What is the maximum height reached by the ball? When is velocity $= 0$

   (b) What is the velocity of the ball when it is 96ft above the ground on its way up? On its way down?

   (c) With what velocity does the ball hit the ground? Position $= 0$

$v(t) = 0$

$v(t) > 0$ $\quad$ $v(t) < 0$

(a) $v(t) = s'(t) = 80 - 32t$

$80 - 32t = 0 \qquad 80 = 32t$

$t = \frac{80}{32} = \frac{10}{4} = \frac{5}{2}$ s

Find height when $t = 5/2 \Rightarrow$ maximum height

$s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = \frac{80 \cdot 5}{2} - \frac{16 \cdot 25}{4} = 40 \cdot 5 - 4 \cdot 25$

$= 200 - 100 = \boxed{100 \text{ ft.}}$

(b) $96 = 80t - 16t^2$

$16t^2 - 80t + 96 = 0$

$16(t^2 - 5t + 6) = 0$

$16(t-2)(t-3) = 0$

$t = 2 \quad t = 3$

On the way up

$t = 2$

$v(2) = 80 - 32(2)$

$= 80 - 64$

$= \boxed{16 \text{ ft./s}}$

On the way down

$t = 3$

$v(3) = 80 - 32(3)$

$= 80 - 96$

$= \boxed{-16 \text{ ft./s}}$

(c) $0 = 80t - 16t^2$

$16t^2 - 80t = 0$

$16t(t-5) = 0$

$\cancel{t=0} \quad t = 5$

$v(5) = 80 - 32(5)$

$= 80 - 160$

$= \boxed{-80 \text{ ft./s}}$