2.6: EXACT DIFFERENTIAL EQUATIONS

Review

- The equation $M(x, y) + N(x, y)y' = 0$ is **exact** if

- How to solve an exact equation:
  1. Define $\Psi_x = M$ and $\Psi_y = N$.
  2. Integrate to find $\Psi(x, y)$.
  3. The equation becomes $\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x, y(x)) = 0$.
  4. Integrate both sides and, if possible, solve for y.

- To find an integrating factor to make an equation exact:
  1. If μ depends only on x,

  2. If μ depends only on y,

Exercise 1

Determine if the equation is exact. If it is exact, find the general solution

$$2x + 4y + (2x - 2y)y' = 0.$$

Exercise 2

Find the value of a for which the equation is exact.

$$3x^2 - axy + 2 + (6y^2 - x^2 + 3)y' = 0.$$

Exercise 3

Determine if the equation is exact. If it is exact, find the general solution.

$$(2 - \ln(x))y' = \frac{y}{x} + 6x, \quad x > 0.$$

Exercise 4

Solve the initial value problem and determine where the solution is valid.

$$(2y - x)y' = y - 2x, \quad y(1) = 3.$$

Exercise 5

Find an integrating factor that makes the following equation exact,

$$1 + \left(\frac{x}{y} - \sin(y) \right) y' = 0.$$

PRACTICE: SOLVING FIRST ORDER ODES

Review

- You do **NOT** need to guess which method to use to solve a 1st order ODE!

- How to determine which method to use:

1. Is the equation **separable**?

2. Is the equation **linear**?

2'. Is it a Bernoulli equation[1]?

3. Is the equation **exact**?

3'. Is it a homogenous equation[2]?

4. If none of the above,[3]

[1] A Bernoulli equation has the form $y' + p(t)y = q(t)y^n$. Not all instructors cover this. You can find examples of Bernoulli equations in Section 2.4 of the textbook, #23–25.

[2] This is NOT the same as the homogeneous linear equations that are covered in Chapter 3. The terminology is confusing. "Homogeneous equation" here refers to a 1st order ODE that can be written in the form $y' = f(\frac{y}{x})$. Not all instructors cover this. You can find examples of these in Section 2.2 of the textbook, #25–31.

[3] Not all instructors cover making an equation exact by using an integrating factor.

Exercise 6

Find the general solution to the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = e^x f - 3xe^{e^x}.$$

Exercise 7

Find the general solution to the differential equation

$$g' + \frac{x^2 + 1}{g} = 0.$$

Exercise 8

Find the general solution to the differential equation

$$(\sin(x) + x^2e^y - 1)y' + y\cos(x) = -2xe^y.$$

Exercise 9

Find the solution to the initial value problem

$$y' = \frac{1-2x}{y}, \qquad y(1) = -2.$$

Exercise 10

Find the general solution to

3.1 SECOND ORDER LINEAR ODES

Review

- A **second order linear ODE** has the form

- A second order linear ODE is **homogeneous** if

- In the next few sections, we are interested in second order homogeneous linear ODEs with **constant coefficients**.

- Process for **solving** a second order homogeneous linear ODE:
  1. Look for solutions of the form $y(t) = e^{rt}$.
  2. Find the characteristic equation.
  3. Find the roots of the characteristic equation.
  4. The general solution is given by
     - Distinct real roots:
     - Complex roots: Section 3.3
     - Repeated real roots: Section 3.4
  5. If you have initial conditions, use them to solve for c_1 and c_2.

Exercise 11

Find the general solution to the differential equation

$$y'' - 4y' + 3y = 0.$$

Exercise 12

Find the general solution to the differential equation

$$2y'' - y' + 5y = 0.$$

Exercise 13

Solve the initial value problem

$$f'' - 2f' - 8f = 0, \quad f(0) = 3, \quad f'(0) = 1.$$