7.2: REVIEW OF MATRICES

Review

- A **matrix** A is a rectangular table of numbers. We typically denote the matrix with a capital letter and its entries with lower case letters. For example, the entry in the 3rd row and 2nd column of A would be denoted a_{32}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

- The **identity matrix** is denoted I and is the matrix that is all zeros, except for along the top-left to bottom-right diagonal, where the entries are all 1. It can have any size but it must be square. For example, the 2×2 and 3×3 cases are

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- A **vector** is just a matrix with either one row or one column.

Basic matrix operations

- To add or subtract two matrices, add or subtract the corresponding entries.
- To multiply a matrix times a scalar (scalar = number), multiply each entry of the matrix by the scalar.

Exercise 1

(a) $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 8 \\ -2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -1 & 8 \\ -1 & 11 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 9 & 3 & -2 \\ 4 & -4 & 1 \\ 0 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 3 & -2 & 1 \\ 2 & -1 & 0 \\ 9 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 5 & -3 \\ 2 & -3 & 1 \\ -9 & 1 & -2 \end{bmatrix}$

(c) $-4 \begin{bmatrix} 2 & 8 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -8 & -32 \\ 4 & -12 \end{bmatrix}$

Matrix multiplication

- To multiply two matrices together, you multiply the rows of the first matrix with the columns of the second matrix.

Exercise 2

(a) $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -1 & 4 & 2 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 12 & 5 \\ 23 & 8 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 0 & -4 \\ 8 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 6 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 38 & 4 \\ 35 & 18 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & -2 & 1 \\ 9 & 0 & 0 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 18 \\ 3 \end{bmatrix}$

Trace

- The trace of a matrix is the sum of its diagonal elements. (The top-left to bottom-right diagonal.)

Exercise 3

Compute the trace of the following matrices.

(a) $\operatorname{tr}\left(\begin{bmatrix} 2 & -9 \\ 4 & -7 \end{bmatrix}\right) = 2 - 7 = -5$

(b) $\operatorname{tr}\left(\begin{bmatrix} 3 & -1 & 0 \\ 9 & 0 & 3 \\ 2 & -2 & -1 \end{bmatrix}\right) = 3 + 0 + (-1) = 2$

Determinant

- The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

- The determinant of a 3×3 matrix is

$$\begin{vmatrix} \overset{+}{a_1} & \overset{-}{a_2} & \overset{+}{a_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Exercise 4

Compute the following determinants.

(a) $\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} = 3 \cdot 3 - (-1) \cdot 2 = 11$

(b) $\begin{vmatrix} -2 & 3 \\ 4 & 9 \end{vmatrix} = (-2)(9) - (3)(4) = -30$

(c) $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 5 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 1 \\ -1 & 3 \end{vmatrix} - 3 \begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} + 4 \begin{vmatrix} -2 & 5 \\ 2 & -1 \end{vmatrix}$

$= 16 - 3(-8) + 4(-8) = 16 + 24 - 32 = \boxed{8}$

Writing a system of linear equations in matrix form

- A common use of matrices and vectors is that we can use them to write a system of linear equations into a compact form.

Exercise 5

Multiply out

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 4 \\ 9 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + 3x_3 \\ 2x_2 + 4x_3 \\ 9x_1 + 3x_3 \end{bmatrix}$$

Exercise 6

Write the following systems of equations into matrix-vector form.

(a)

$$\begin{aligned} 3x_1 - x_2 &= 7 \\ 6x_1 + 3x_2 &= 5 \end{aligned}$$

$$\begin{bmatrix} 3 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

(b)

$$
\begin{aligned}
4x_1 - 3x_2 + x_3 &= 0 \\
0x_1 + 3x_2 + 4x_3 &= 4 \\
-x_2 &= 2
\end{aligned}
$$

$$
\begin{bmatrix} 4 & -3 & 1 \\ 0 & 3 & 4 \\ 0 & -1 & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}
$$

(c)

$$
\begin{aligned}
x_3 - x_2 &= 7 \\
3x_2 + x_1 &= 4 \\
2x_1 - 3x_3 + 2x_2 &= 3
\end{aligned}
$$

$$
\begin{bmatrix} 0 & -1 & 1 \\ 1 & 3 & 0 \\ 2 & 2 & -3 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}
$$

7.1: INTRODUCTION TO SYSTEMS OF 1ST-ORDER ODES

Review

- A **system of differential equations** is just a few differential equations. The interesting part is that they might be coupled (that is, the solution to one equation depends on the solution to another one). For example,

$$u' = 3v + u$$
$$v' = -v + 4u$$

- A system of first-order differential equations is **linear** if they can be written in the form

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + ... + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + ... + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + ... + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

$$P(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & & p_{nn}(t) \end{bmatrix}$$

  If we write this into matrix form, it looks like

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$$

$$\vec{g}(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

- A system of linear first-order differential equations is **homogeneous** if the g_i terms are all 0. (i.e., if $\mathbf{g}(t) = \mathbf{0}$.)

- A system of differential equations can also have initial conditions. For example, the above system could have the initial conditions

$$x_1(0) = a_1, \quad x_2(0) = a_2, \quad ..., \quad x_n(0) = a_n,$$

  which can also be written as $\mathbf{x}(0) = \mathbf{a}$.

- You can convert a higher order differential equation into a system of first-order differential equations

Exercise 7

For the following systems of ODEs, determine if they are linear or nonlinear. If they are linear, then also determine if they are homogeneous or nonhomogeneous. It if is linear, write it into matrix-vector form.

(a)

$$x_1 - x_2 = 3t$$
$$x_1^2 + x_2 = \sin(t)$$

nonlinear

(b)

$$3t^3 x_1 + x_2 - t x_3 = \cos(t)$$
$$x_1 - x_3 = 0$$
$$x_2 + \tan(t) x_3 = 0$$

$$\begin{bmatrix} 3t^3 & 1 & -t \\ 1 & 0 & -1 \\ 0 & 1 & \tan(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos(t) \\ 0 \\ 0 \end{bmatrix}$$

linear, nonhomogeneous

(c)

$$x_1 - x_2 x_3 = 7$$
$$x_2 - \cos(x_3) = 4t$$
$$3x_1 + (t^6 + 4) x_3 = 0$$

nonlinear

(d)

$$\cos(t) x_1 - x_3 = 0$$
$$x_1 - 5\ln(t) x_2 = 5t x_3 \rightarrow x_1 - 5\ln(t) x_2 - 5t x_3 = 0$$
$$x_2 + x_3 = 0$$

$$\begin{bmatrix} \cos(t) & 0 & -1 \\ 1 & -5\ln(t) & -5t \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

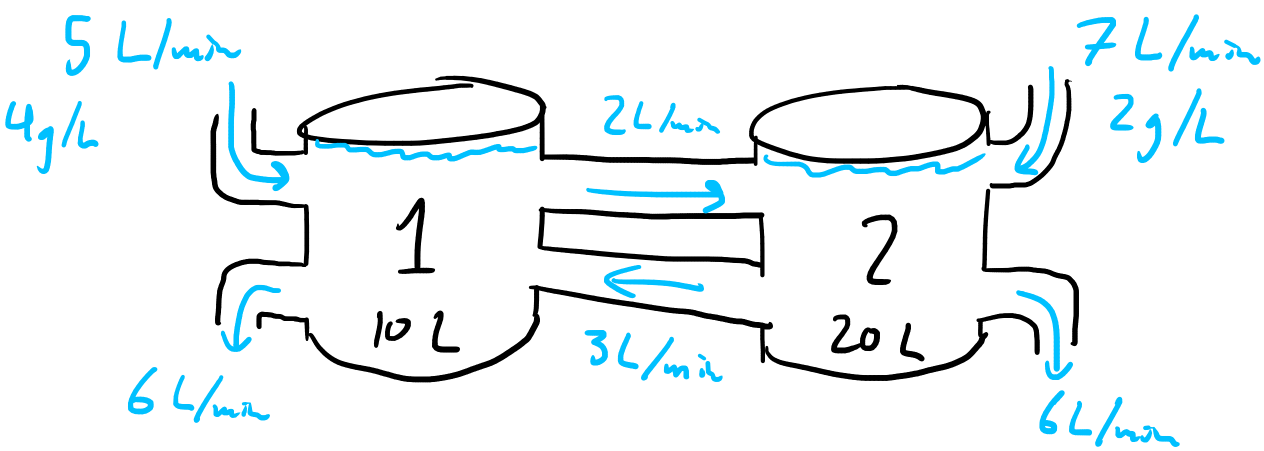
linear, homogeneous

Exercise 8

Suppose we have two tanks. Tank 1 holds 10 L and tank 2 holds 20 L.

- Salt water containing 4 g/L of salt is flowing into tank 1 at a rate of 5 L/min.
- Salt water containing 2 g/L of salt is flowing into tank 2 at a rate of 7 L/min.
- Salt water flows out of tank 1 at a rate of 8 L/min, of which 2 L/min flows into tank 2 and the rest leaves the system.
- Salt water flows out of tank 2 at a rate of 9 L/min, of which 3 L/min flows into tank 1 and the rest leaves the system.

Both tanks initially start with 40 g of salt. Write down an initial value problem that models this system. Write your answer into matrix-vector form.



$S_1(t)$ = grams of salt in tank 1.

$S_2(t)$ = ____________________ 2.

$$\frac{dS_1}{dt} = +\left(5\,\frac{L}{min}\right)\left(4\,\frac{g}{L}\right) + \left(3\,\frac{L}{min}\right)\left(\frac{S_2}{20}\,\frac{g}{L}\right) - \left(8\,\frac{L}{min}\right)\left(\frac{S_1}{10}\,\frac{g}{L}\right)$$

$$\frac{dS_2}{dt} = +\left(7\,\frac{L}{min}\right)\left(2\,\frac{g}{L}\right) + \left(2\,\frac{L}{min}\right)\left(\frac{S_1}{10}\,\frac{g}{L}\right) - \left(9\,\frac{L}{min}\right)\left(\frac{S_2}{20}\,\frac{g}{L}\right)$$

$$S_1(0) = 40\,g \qquad S_2(0) = 40\,g$$

$$\vec{S} = \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix}$$

$$\boxed{\vec{S}\,' = \begin{bmatrix} 20 \\ 14 \end{bmatrix} + \begin{bmatrix} \frac{-8}{10} & \frac{3}{20} \\ \frac{2}{10} & \frac{-9}{20} \end{bmatrix} \vec{S}, \quad \vec{S}(0) = \begin{bmatrix} 40 \\ 40 \end{bmatrix}}$$

Exercise 9

Suppose we initially have 150 rabbits that reproduce at a rate proportional to the current population. There are ten wolves that live nearby. The wolves eat the rabbits at a rate proportional to the product of the population of rabbits and the population of wolves. The wolves reproduce at a rate proportional to the number of rabbits they eat. Write down an initial value problem that models this system.

$R(t) = \#$ rabbits at time t

$W(t) = \#$ wolves at time t

$$\boxed{\begin{aligned} \frac{dR}{dt} &= k_1 R - k_2 RW, \qquad R(0) = 150 \\ \frac{dW}{dt} &= k_3 RW, \qquad W(0) = 10 \end{aligned}}$$

Exercise 10

Write the following differential equation as a system of differential equations.

$$y'' - 5y' + 6y = \cos(t)$$

$$\downarrow$$

$x_1 = y$

$x_2 = y' = x_1'$

$$x_2' - 5x_2 + 6x_1 = \cos(t)$$

$$\begin{cases} x_1' = x_2 \\ x_2' = -6x_1 + 5x_2 + \cos(t) \end{cases}$$

$$\boxed{\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}}$$

Exercise 11

Write the following initial value problem as a system of differential equations.

$$g''' - t^2 g'' + \sin(t) g' - g = 4t, \quad g(0) = 3, \quad g'(0) = 4, \quad g''(0) = 0.$$

$x_1(0) = 3 \quad x_2(0) = 4 \quad x_3(0) = 0$

$x_1 = g$

$x_2 = g' = x_1'$

$x_3 = g'' = x_2'$

$x_3' - t^2 x_3 + \sin(t) x_2 - x_1 = 4t$

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = x_1 - \sin(t) x_2 + t^2 x_3 + 4t, \end{cases} \qquad \begin{aligned} x_1(0) &= 3 \\ x_2(0) &= 4 \\ x_3(0) &= 0 \end{aligned}$$

$$\boxed{\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -\sin(t) & t^2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 4t \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}}$$

Exercise 12

Write the following initial value problem as a system of differential equations.

$$z^{(4)} - tz'' + z = 7t^3 + \cos(t), \quad z(2) = 1, \quad z'(2) = -1, \quad z''(2) = 6, \quad z'''(2) = 4.$$

$x_1(2) = 1 \quad x_2(2) = -1 \quad x_3(2) = 6 \quad x_4(2) = 4$

$x_1 = z$

$x_2 = z' = x_1'$

$x_3 = z'' = x_2'$

$x_4 = z''' = x_3'$

$x_4' - t x_3 + x_1 = 7t^3 + \cos(t)$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = -x_1 + t x_3 + 7t^3 + \cos(t) \end{cases} \qquad \begin{aligned} x_1(2) &= 1 \\ x_2(2) &= -1 \\ x_3(2) &= 6 \\ x_4(2) &= 4 \end{aligned}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & t & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7t^3 + \cos(t) \end{bmatrix}$$

$$\vec{x}(2) = \begin{bmatrix} 1 \\ -1 \\ 6 \\ 4 \end{bmatrix}$$

7.3: LINEAR INDEPENDENCE, EIGENVALUES, AND EIGENVECTORS

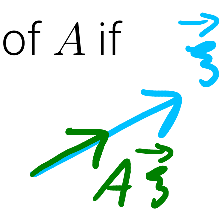
Review

- Two vectors are **linearly dependent** if one can is a scalar multiple of the other.
- More generally, n vectors $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ are linearly dependent if there exist constants c_1, ..., c_n such that

$$c_1\mathbf{x}^{(1)} + \dots + c_n\mathbf{x}^{(n)} = \mathbf{0}.$$

- If a set of vectors is not linearly dependent, then we say they are **linearly independent**.
- λ is an **eigenvalue** and ξ is a corresponding **eigenvector** of A if

$$A\xi = \lambda\xi.$$



- How to find eigenvalues/vectors for 2x2 matrices
  1. Find the characteristic equation: $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$.
  2. Solve the characteristic equation to find the eigenvalues.
  3. Nonzero solutions to $(A - \lambda I)\xi = 0$ are the eigenvectors.

Exercise 13

For each pair of vectors, are they linearly dependent or independent?

(a) $\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

dependent

(b) $\left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$

independent

(c) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

dependent

(d) $\left\{ \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$

independent

Exercise 14

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$.

Characteristic equation: $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 4)(\lambda - 2) = 0$$

$$\lambda = 2, 4$$ ← eigenvalues

eigenvector for $\lambda = 2$:

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\vec{\xi} = \vec{0} \qquad 3\xi_1 - \xi_2 = 0 \Rightarrow \xi_2 = 3\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ 3\xi_1 \end{bmatrix} = \xi_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

eigenvector for any nonzero ξ_1

eigenvector for $\lambda = 4$:

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\vec{\xi} = \vec{0} \Rightarrow \xi_1 - \xi_2 = 0 \Rightarrow \xi_1 = \xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \xi_2 \end{bmatrix} = \xi_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\left\{2, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}, \left\{4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}$$

Exercise 15

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

characteristic equation: $\lambda^2 - 2\lambda + 5 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm \frac{4i}{2} = \underbrace{1 \pm 2i}_{\text{eigenvalues}}$$

eigenvector for $\lambda = 1 + 2i$:

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1+2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix} \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow 4\xi_1 - (2+2i)\xi_2 = 0$$

$$\xi_1 = \frac{2+2i}{4} \xi_2 = \frac{1+i}{2} \xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \frac{1+i}{2} \xi_2 \\ \xi_2 \end{bmatrix} = \cancel{\xi_2}\, 2 \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

eigenvector for $\lambda = 1-2i$:

$$\begin{bmatrix} 3-(1-2i) & -2 \\ 4 & -1-(1-2i) \end{bmatrix} \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow (2+2i)\xi_1 - 2\xi_2 = 0$$

$$\xi_2 = \frac{2+2i}{2}\xi_1 = (1+i)\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ (1+i)\xi_1 \end{bmatrix} = \overset{1}{\cancel{\xi_1}} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$\boxed{\left\{ 1+2i, \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \right\}, \left\{ 1-2i, \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \right\}}$$

Exercise 16

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$.

characteristic eq: $\lambda^2 - 0 \cdot \lambda - 4 = 0 \Rightarrow \lambda^2 = 4$

$$\lambda = \underbrace{\pm 2}_{\text{eigenvalues}}$$

eigenvector for $\lambda = -2$:

$$\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \sqrt{3}\,\xi_1 + \xi_2 = 0 \Rightarrow \xi_2 = -\sqrt{3}\,\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ -\sqrt{3}\,\xi_1 \end{bmatrix} = \cancelto{1}{\xi_1} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

eigenvector for $\lambda = 2$:

$$\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow -\xi_1 + \sqrt{3}\,\xi_2 = 0$$

$$\xi_1 = \sqrt{3}\,\xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}\,\xi_2 \\ \xi_2 \end{bmatrix} = \cancelto{1}{\xi_2} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

$$\boxed{\left\{ -2, \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} \right\}, \left\{ 2, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \right\}}$$