WIR#4

Section 2.3

x^2 vs. 2^x

Power Function — Exp Function

$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$

$\frac{d}{dx}(2^x) = \ln 2 \cdot 2^x$

- If $f(x) = k$, where k is a constant, then $f'(x) = 0$.
- If $f(x) = x^n$ then $f'(x) = nx^{n-1}$. ← Power Rule
- If $f(x) = b^x$ then $f'(x) = \ln b \cdot b^x$ ← Exp. Rule
- If $f(x) = e^x$ then $f'(x) = e^x$ Special case of
- If $f(x) = \log_b x$ then $f'(x) = \frac{1}{\ln b} \cdot \frac{1}{x} = \frac{1}{x \ln b}$
- $\ln x = \log_e(x)$ If $f(x) = \ln x$ then $f'(x) = \frac{1}{x}$ special case of
- If $h(x) = k \cdot f(x)$ where k is a constant, then $h'(x) = k \cdot f'(x)$
- If $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$
- **Marginal Business Functions**:

  Take the derivative

  - If $C(x)$ is the total cost of producing x items, then $C'(x)$ is the **Marginal Cost Function**.
  - If $R(x)$ is the total revenue from selling x items, then $R'(x)$ is the **Marginal Revenue Function**.
  - If $P(x)$ is the total profit from making and selling x items, then $P'(x)$ is the **Marginal Profit Function**.
- The Marginal Business Functions can be used to **approximate** the profit, revenue, or cost of the **next (i.e. single) item**. using the derivative!
  - The **approximate** cost from making the n^{th} item is $C'(n-1)$.
  - The **approximate** revenue from selling the n^{th} item is $R'(n-1)$.
  - The **approximate** profit from making and selling the n^{th} item is $P'(n-1)$.
- The Business Functions can be used to find the **exact** profit, revenue, or cost of **a single item**. using the original function!
  - The **exact** cost of making the n^{th} item is $C(n) - C(n-1)$.
  - The **exact** revenue of selling the n^{th} item is $R(n) - R(n-1)$.
  - The **exact** profit of making and selling the n^{th} item is $P(n) - P(n-1)$.
- The *nDeriv*(command can be used on your calculator to approximate the value of $f'(a)$. It can be found by hitting MATH and then 8. The format for the command is $nDeriv(Y_1, X, a)$ (assuming the function is typed into Y_1).

  Note: The command may instead appear symbolically on your homescreen. If this is the case, you would need to enter the same inputs (Y_1, X, a) into the appropriate boxes in the symbolic notation.

take the derivative of what is inside the parenthesis

1. Evaluate the following: $\frac{d}{dx}\left(3x^2 + 2x^1 - 4e^x + 5\ln(x) + e - 4\right)$

$= 3\cdot 2x^{2-1} + 2\cdot 1x^{1-1} - 4\cdot e^x + 5\cdot\frac{1}{x} + 0 - 0$

$= \boxed{6x + 2 - 4e^x + \frac{5}{x}}$

2. Find $\frac{dy}{dx}$ if $y = 4\sqrt{x} - 4x^2 + 2^x - 4\ln(x) - \frac{5}{x^8} + \log_7(x)$

$= 4x^{1/2} - 4x^2 + 2^x - 4\ln(x) - 5x^{-8} + \log_7(x)$

$= 4\cdot\frac{1}{2}x^{\frac{1}{2}-1} - 4\cdot 2x^{2-1} + \ln 2\cdot 2^x - 4\cdot\frac{1}{x} - 5\cdot(-8x^{-8-1}) + \frac{1}{\ln 7}\cdot\frac{1}{x}$

$= \boxed{2x^{-1/2} - 8x + \ln 2\cdot 2^x - \frac{4}{x} + 40x^{-9} + \frac{1}{\ln 7}\cdot\frac{1}{x}}$

3. Find $h'(x)$ if $h(x) = (3x^2 + x^5)\left(\sqrt[5]{x^3} - 4\right)$

$\sqrt[5]{x^3} = x^{3/5}$

FOIL!

$\sqrt[5]{x^3} = (x^3)^{1/5} = x^{3/5}$

$= 3x^2\cdot x^{3/5} - 3x^2\cdot 4 + x^5\cdot x^{3/5} - 4\cdot x^5$

$= 3x^{2+\frac{3}{5}} - 12x^2 + x^{5+\frac{3}{5}} - 4x^5$

$= 3x^{13/5} - 12x^2 + x^{28/5} - 4x^5$

$h'(x) = 3\cdot\frac{13}{5}x^{\frac{13}{5}-1} - 12\cdot 2x^{2-1} + \frac{28}{5}x^{\frac{28}{5}-1} - 4\cdot 5x^{5-1}$

$\boxed{h'(x) = \frac{39}{5}x^{8/5} - 24x + \frac{28}{5}x^{23/5} - 20x^4}$

4. Find $k'(x)$ if $k(x) = \dfrac{\sqrt[7]{x^3} + 4x^5 - \frac{1}{x^2}}{\sqrt[3]{x^2}}$

$= \dfrac{x^{3/7} + 4x^5 - x^{-2}}{x^{2/3}} = \dfrac{x^{3/7}}{x^{2/3}} + \dfrac{4x^5}{x^{2/3}} - \dfrac{x^{-2}}{x^{2/3}}$

$= x^{\frac{3}{7}-\frac{2}{3}} + 4x^{5-\frac{2}{3}} - x^{-2-\frac{2}{3}}$

$= x^{-\frac{5}{21}} + 4x^{13/3} - x^{-8/3}$

$k'(x) = \frac{-5}{21}x^{\frac{-5}{21}-1} + 4\cdot\frac{13}{3}x^{\frac{13}{3}-1} - \left(-\frac{8}{3}x^{-\frac{8}{3}-1}\right)$

$\boxed{k'(x) = -\frac{5}{21}x^{\frac{-26}{21}} + \frac{52}{3}x^{\frac{10}{3}} + \frac{8}{3}x^{-11/3}}$

5. Determine the value(s) of x for which the line tangent to the graph of $f(x) = 4x^2 - 7x + 5$ is parallel to the line $y = 10x - 5$.

① For the lines to be parallel, we need the slopes to be equal.

② The slope of the given line is 10.

③ The slope of the line tangent to f(x) is given by f'(x).

$f'(x) = 4 \cdot 2x - 7 \cdot 1 + 0$
$= 8x - 7$

④ Thus, we need $f'(x) = 10$

$8x - 7 = 10$
$8x = 17$
$\boxed{x = \frac{17}{8}}$

6. The total profit function for a particular storage box is given by $P(x) = \overbrace{-0.0012x^2 + 10.45x - 500}^{y_1}$. (in dollars) when x storage boxes are sold.

(a) Find the exact profit realized from the sale of the 100th storage box.

single item

$P(100) - P(99)$

$= 533 - 522.7888$

$\approx \boxed{\$10.21}$

(b) Use the marginal profit function to estimate the profit realized from the sale of the 100th storage box.

n single item

Marginal Profit $= P'(x) = -0.0012 \cdot 2x + 10.45 - 0$
$= -0.0024x + 10.45$

Profit from the 100th box $\approx P'(100-1) = P'(99) = -0.0024(99) + 10.45$
$\approx \boxed{\$10.21}$

7. The price-demand function for a particular bag of coffee is given by $p = -\frac{2}{15}x + 10$, where x bags are sold at a unit price of $\$p$.

   (a) Find the revenue function.

   Revenue = (quantity)(unit price)

   $R(x) = x \cdot p$

   $R(x) = x\left(-\frac{2}{15}x + 10\right)$

   $\boxed{R(x) = -\frac{2}{15}x^2 + 10x}$

   (b) Find the exact revenue of selling 30 bags of coffee.

   not a single item, we are talking about all 30 bags!

   $R(30) = \frac{-2}{15}(30)^2 + 10(30)$

   $= \boxed{\$180}$

   (c) Use marginal analysis to approximate the revenue of selling the 30th bag.

   single item

   Marginal Revenue $= R'(x) = \frac{-2}{15} \cdot 2x + 10$

   $= \frac{-4}{15}x + 10$

   Revenue from the 30th bag $\approx R'(30-1) = R'(29) = -\frac{4}{15}(29) + 10 \approx \boxed{\$2.27}$

   OR nDeriv$\left(-\frac{2}{15}x^2 + 10x, x, 29\right) = \frac{d}{d\boxed{x}}\left(\boxed{-\frac{2}{15}x^2 + 10x}\right)_{x=\boxed{29}}$

   $\approx \boxed{\$2.27}$

Section 2.4

- If $h(x) = f(x) \cdot g(x)$ then $h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
- If $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$

$h(x) = F \cdot S \quad h'(x) = F \cdot S' + S \cdot F'$

$h(x) = \frac{T}{B} \quad h'(x) = \frac{B \cdot T' - T \cdot B'}{B^2}$

9. Find the derivative of each of the following functions:

   (a) $f(x) = \underbrace{(2x^2 - 5x)}_{F}\underbrace{\left(4x^4 - \sqrt[5]{x^3} + 8\right)}_{S}$ $\quad x^{3/5}$

   $F = 2x^2 - 5x \qquad S = 4x^4 - x^{3/5} + 8$

   $F' = 4x - 5 \qquad S' = 16x^3 - \frac{3}{5}x^{-2/5}$

   $$\boxed{f'(x) = \underbrace{(2x^2-5x)}_{F}\underbrace{\left(16x^3 - \frac{3}{5}x^{-2/5}\right)}_{S'} + \underbrace{(4x^4 - x^{3/5} + 8)}_{S}\underbrace{(4x-5)}_{F'}}$$

$\frac{d}{dx}\left(\frac{T}{B}\right) = \frac{B\cdot T' - T\cdot B'}{B^2}$

(b) $g(x) = \dfrac{\overbrace{3x^4 - 2^x}^{T}}{\underbrace{8x^7 - 5}_{B}}$

$T = 3x^4 - 2^x$ $\quad B = 8x^7 - 5$

$T' = 12x^3 - \ln 2\cdot 2^x$ $\quad B' = 56x^6$

$$\boxed{g'(x) = \frac{\overset{B}{(8x^7-5)}\overset{T'}{(12x^3-\ln 2\cdot 2^x)} - \overset{T}{(3x^4-2^x)}\overset{B'}{(56x^6)}}{\underset{B}{(8x^7-5)^2}}}$$

(c) $m(t) = \left(\underbrace{2t^2 - \frac{1}{t} + \ln(t)}_{F}\right)\left(\underbrace{e^t + 7\log_3(t)}_{S}\right)$ $\quad t^{-1}$

$F = 2t^2 - t^{-1} + \ln t$ $\quad S = e^t + 7\log_3(t)$

$F' = 4t + 1t^{-2} + \frac{1}{t}$ $\quad S' = e^t + 7\cdot\frac{1}{\ln 3}\cdot\frac{1}{t}$

$$\boxed{m'(t) = \overset{F}{(2t^2 - t^{-1} + \ln(t))}\overset{S'}{\left(e^t + 7\cdot\frac{1}{\ln 3}\cdot\frac{1}{t}\right)} + \overset{S}{(e^t + 7\log_3(t))}\overset{F'}{\left(4t + t^{-2} + \frac{1}{t}\right)}}$$

$7\cdot\frac{1}{\ln 3}\cdot\frac{1}{t} = \frac{7}{\ln 3\cdot t} = \frac{7}{t\cdot\ln 3}$

(d) $C(x) = \dfrac{\overbrace{x^2 + 7x - 4}^{T}}{\underbrace{4^x - 5x\ln(x)}_{B}}$

$T = x^2 + 7x - 4$ $\quad B = 4^x - \overset{F}{5x}\,\overset{S}{\ln(x)}$

$T' = 2x + 7$ $\quad B' = \ln 4\cdot 4^x - \left(\underset{F}{5x}\cdot\underset{S'}{\frac{1}{x}} + \underset{S}{\ln(x)}\cdot\underset{F'}{5}\right)$

$$\boxed{C'(x) = \frac{\overset{B}{(4^x - 5x\ln(x))}\overset{T'}{(2x+7)} - \overset{T}{(x^2+7x-4)}\overset{B'}{\left(\ln 4\cdot 4^x - \left(5x\cdot\frac{1}{x} + \ln(x)\cdot 5\right)\right)}}{\underset{B}{(4^x - 5x\ln(x))^2}}}$$

10. Find the equation of the line tangent to the graph of $f(x) = \underbrace{4x^2}_{F}\underbrace{e^x}_{S}$ at $x = 1$.

$y = mx + b$ $\quad y - y_1 = m(x - x_1)$

① slope of the tangent line at $x=1$:

$f'(x) = \underset{F}{(4x^2)}\underset{S'}{(e^x)} + \underset{S}{(e^x)}\underset{F'}{(8x)}$

$f'(1) = 4(1)^2\cdot e^1 + e^1\cdot 8\cdot 1$

$= 4e + 8e$

$= 12e$ ← slope of the tangent line

② we find $f(1)$ to give us the y-coord. of the point.

$f(1) = 4(1)^2 e^1 = 4e$

③ Now we find the eq. of the line

$(1, 4e)$ $\quad m = 12e$

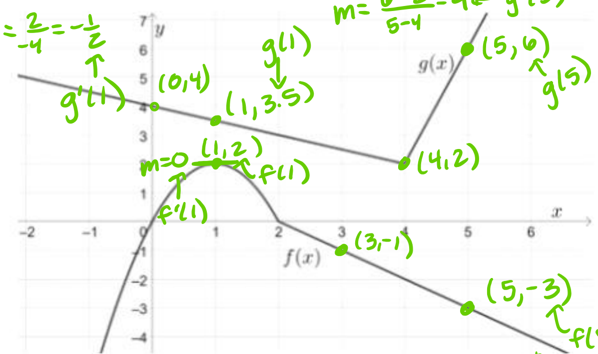
$y - 4e = 12e(x - 1)$

$y - 4e = 12ex - 12e$

$+4e \qquad +4e$

$\boxed{y = 12ex - 8e}$

11. Given the graphs of $f(x)$ and $g(x)$ below, determine the following:



(a) $k'(5)$ if $k(x) = \underbrace{(x^4 - f(x))}_{F(x)}\underbrace{(g(x) - 2x)}_{S(x)}$

$F(x) = x^4 - f(x)$

$F'(x) = 4x^3 - f'(x)$

$S(x) = g(x) - 2x$

$S'(x) = g'(x) - 2$

$x=5$

$F(5) = 5^4 - f(5) = 625 - (-3) = 628$

$F'(5) = 4(5)^3 - f'(5) = 500 - (-1) = 501$

$S(5) = g(5) - 2(5) = 6 - 10 = -4$

$S'(5) = g'(5) - 2 = 4 - 2 = 2$

$k'(x) = F(x)\cdot S'(x) + S(x)\cdot F'(x)$

$k'(5) = F(5)\cdot S'(5) + S(5)\cdot F'(5)$

$= (628)(2) + (-4)(501)$

$= \boxed{-748}$

(b) $h'(1)$ if $h(x) = \dfrac{\overbrace{10 + f(x)}^{T(x)}}{\underbrace{3x^4 - g(x)}_{B(x)}}$

$T(x) = 10 + f(x)$

$T'(x) = f'(x)$

$B(x) = 3x^4 - g(x)$

$B'(x) = 12x^3 - g'(x)$

$x = 1$

$T(1) = 10 + f(1) = 10 + 2 = 12$

$T'(1) = f'(1) = 0$

$B(1) = 3(1)^4 - g(1) = 3 - 3.5 = -0.5$

$B'(1) = 12(1)^3 - g'(1) = 12 - (-\frac{1}{2}) = 12.5$

$$h'(x) = \frac{B(x)\cdot T'(x) - T(x)\cdot B'(x)}{(B(x))^2}$$

$$h'(1) = \frac{B(1)\cdot T'(1) - T(1)\cdot B'(1)}{(B(1))^2} = \frac{(-0.5)(0) - (12)(12.5)}{(-0.5)^2} = \boxed{-600}$$

T

12. For what value(s) of x is the line tangent to the graph of $f(x) = \dfrac{x^2}{\underbrace{x^2 - x + 1}_{B}}$ horizontal?

$\Rightarrow$ where is $f'(x) = 0$?

$$f'(x) = \frac{(x^2 - x + 1)(2x) - (x^2)(2x-1)}{(x^2 - x + 1)^2}$$

B T' T B'

B

$$= \frac{\cancel{2x^3} - 2x^2 + 2x - \cancel{2x^3} + x^2}{(x^2 - x + 1)^2} = \frac{-x^2 + 2x}{(x^2 - x + 1)^2}$$

$f'(x) = 0$ when

$-x^2 + 2x = 0$

$x(-x + 2) = 0$

$\boxed{x = 0}$ $\quad -x + 2 = 0$

$\boxed{2 = x}$

13. The cost function (in dollars) for a company that makes coffee is given by $C(x) = \dfrac{\overbrace{10x^{5/2}}^{T}}{\underbrace{x^2+x+2}_{B}}$ when x pounds of coffee are made. Find (and interpret) $\underbrace{\text{the marginal cost when 4 pounds of coffee are made.}}_{C'(4)?}$

$$C'(x) = \frac{\overset{B}{(x^2+x+2)}\overset{T'}{(\cancel{10}^{5} \cdot \frac{5}{\cancel{2}} x^{3/2})} - \overset{T}{(10x^{5/2})}\overset{B'}{(2x+1)}}{\underset{B}{(x^2+x+2)^2}}$$

$$C'(4) = \frac{(4^2+4+2)(25(4)^{3/2}) - (10(4^{5/2}))(2(4)+1)}{(4^2+4+2)^2} = \frac{(22)(200)-(320)(9)}{22^2} \approx \boxed{\$3.14 \text{ / pound}}$$

When the company makes 4 pounds of coffee, their cost is increasing by \$3.14/pound

14. If $f(x) = \dfrac{\overbrace{\ln(x)}^{T}}{\underbrace{x^5}_{B}}$, what is $f'(e)$?

$$f'(x) = \frac{\overset{B}{\cancel{x^5}^{4}} \cdot \overset{T'}{\frac{1}{\cancel{x}}} - \overset{T}{\ln(x)} \cdot \overset{B'}{5x^4}}{\underset{B}{(x^5)^2}} = \frac{x^4 - 5x^4\ln(x)}{x^{10}}$$

$$f'(e) = \frac{e^4 - 5e^4\overbrace{\ln(e)}^{1}}{e^{10}} = \frac{e^4-5e^4}{e^{10}} = \frac{-4\cancel{e^4}}{e^{\cancel{10}\,6}} = \boxed{\frac{-4}{e^6}}$$