Week 10

Monday, October 28, 2024 7:22 PM

TEXAS A&M UNIVERSITY
Math Learning Center

Math 151 - Fall 2024
Week-In-Review 10

Math 151
Week-In-Review 10

4.2, 4.3
Todd Schrader

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Problem Statements

1. Determine the value(s) of c that satisfies the conclusion of the Mean Value Theorem for the given function on the interval.

   (a) $f(x) = \frac{1}{x}$ on [1,5]

Avg. Rate of Change: $\frac{f(5)-f(1)}{5-1} = \frac{\frac{1}{5}-1}{4} = \frac{-4/5}{4} = -1/5$

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$f'(c) = \frac{-1}{c^2}$

$\frac{-1}{c^2} = \frac{-1}{5}$

$c^2 = 5$

$c = \pm\sqrt{5}$

$\boxed{c = \sqrt{5}}$

   (b) $f(x) = e^{x/2}$ on [2,4]

$\frac{f(4)-f(2)}{4-2} = \frac{e^{4/2}-e^{2/2}}{2} = \frac{e^2-e}{2}$

$f'(x) = e^{x/2}\cdot\frac{1}{2}$

$f'(c) = \frac{e^{c/2}}{2} = \frac{e^2-e}{2}$

$e^{c/2} = e^2-e$

$\ln(e^{c/2}) = \ln(e^2-e)$

$c/2 = \ln(e^2-e)$

$\boxed{c = 2\ln(e^2-e)}$

Typo

2. Let $f(x) = (x-2)^{-2}$. Show there is no value of c in $(1,4)$ such that $f'(c) = \frac{f(4)-f(1)}{4-1}$. Does this contradict the Mean Value Theorem?

$\frac{\frac{1}{(4-2)^2} - \frac{1}{(1-2)^2}}{4-1} = \frac{\frac{1}{4}-\frac{1}{1}}{3} = \frac{-\frac{3}{4}}{3/1} = \frac{-1}{4}$

$f'(x) = -2(x-2)^{-3}\cdot(1)$

$f'(c) = \frac{-2}{(x-2)^3} = \frac{-1}{4}$

$8 = (x-2)^3$

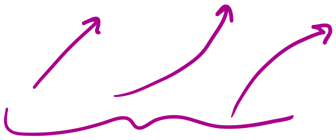
$2 = x-2$

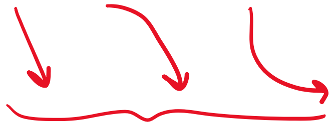
$x = 4$

Conditions of MVT don't apply, since $f(x)$ is not continuous on [1,4].

3. A few understanding questions:

(a) What does an increasing function look like? What does a decreasing function look like?





(b) What does it mean for a function to be increasing/decreasing?

Slopes positive

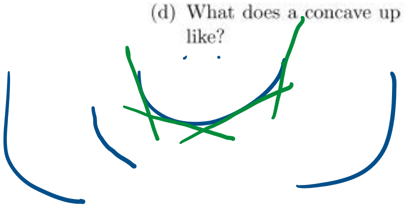
Slopes are negative

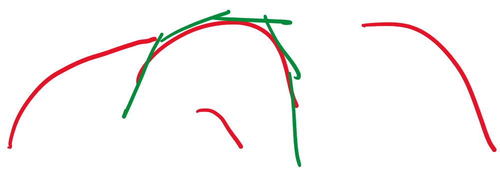
(c) How do we determine when a function is increasing/decreasing?

$f'(x) > 0$

$f'(x) < 0$

(d) What does a concave up function look like? What does a concave down function look like?





(e) What does it mean for a function to be concave up/down?

Slopes are becoming more positive
(Slopes are Increasing)
$\updownarrow$
$f'(x)$ are Increasing

Slopes are becoming more negative
(Slopes are Decreasing)

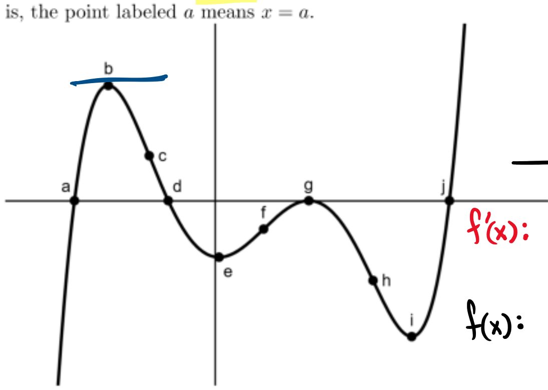
(f) How do we determine when a function is concave up/down?

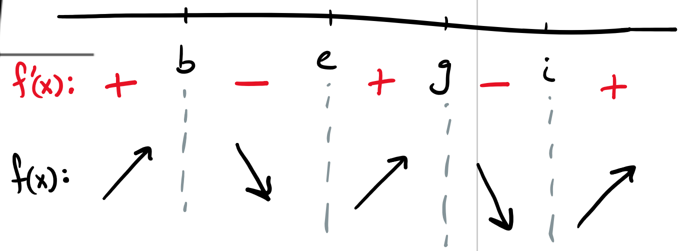
$f''(x) > 0$

$f''(x) < 0$

Critical #s of $f(x)$: $f'(x)$ DNE or $f'(x) = 0$

4. Consider the graph of $f(x)$ below. Assume each label corresponds with an $x-$value. That is, the point labeled a means $x = a$.





(a) Determine the critical numbers of $f(x)$, that, is where $f'(x)$ DNE or $f'(x) = 0$.

$f'(x) = 0$: $\boxed{x = b,\ e,\ g,\ i}$

(b) Determine the intervals where $f(x)$ is increasing/decreasing.

$f(x)$ Increasing : $(-\infty, b)$, (e, g), (i, ∞)

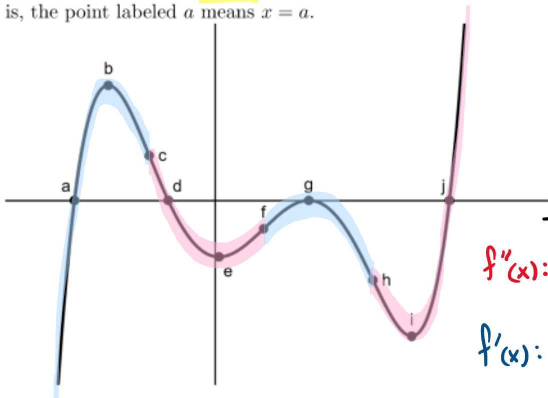
$f(x)$ Decreasing : (b, e), (g, i)

(c) Determine the location of any local extrema of $f(x)$.

Local Maximum : @ $x = b$, $x = g$

Local Minimum : @ $x = e$, $x = i$

5. Consider the graph of $f(x)$ below. Assume each label corresponds with an $x-$value. That is, the point labeled a means $x = a$.



c f h

$f''(x)$: $-$ | c | $+$ | f | $-$ | h | $+$

$f'(x)$: $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$

$f(x)$: $\cap$ | $\cup$ | $\cap$ | $\cup$

(a) Determine the critical numbers of $f'(x)$, that is, where $f''(x)$ DNE or $f''(x) = 0$.

$f''(x) = 0$: $\boxed{x = c, f, h}$

(b) Determine the intervals where $f(x)$ is concave up/down.

Concave Up: $(c, f), (h, \infty)$

Concave Down: $(-\infty, c), (f, h)$

(c) Determine the location of any inflection points of $f(x)$.

@ $\boxed{x = c, f, h}$

(d) Determine the intervals where $f'(x)$ is increasing/decreasing.

$f'(x)$ Increasing: $(c, f), (h, \infty)$ $\Rightarrow f''(x) > 0 \Rightarrow f(x)$ Concave Up

$f'(x)$ Decreasing: $(-\infty, c), (f, h)$

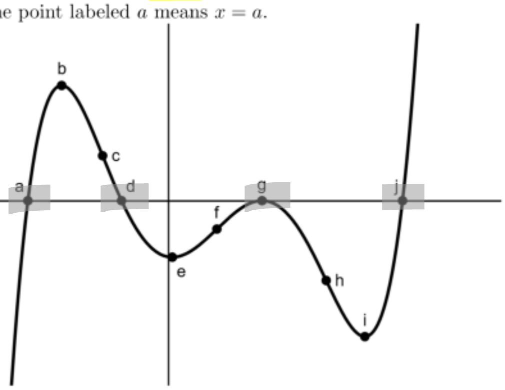
(e) Determine the location of any local extrema of $f'(x)$

$f'(x)$ Local Minimum @ $x = c$, $x = h$

$f'(x)$ Local Maximum @ $x = f$

6. Consider the graph of $f'(x)$ below. Assume each label corresponds with an $x-$value. That is, the point labeled a means $x = a$.

0 =



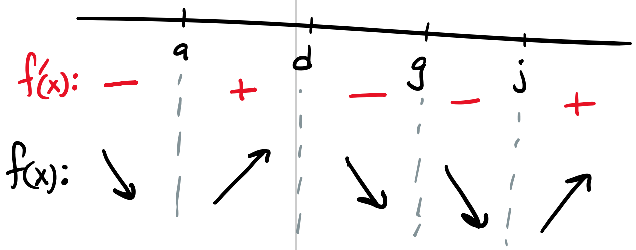
(a) Determine the critical numbers of $f(x)$, that, is where $f'(x)$ DNE or $f'(x) = 0$.

$f'(x) = 0$ $\boxed{x = a,\ d,\ g,\ j}$

(b) Determine the intervals where $f(x)$ is increasing/decreasing.

$f(x)$ Increasing: $(a, d), (j, \infty)$

$f(x)$ Decreasing: $(-\infty, a), (d, g), (g, j)$

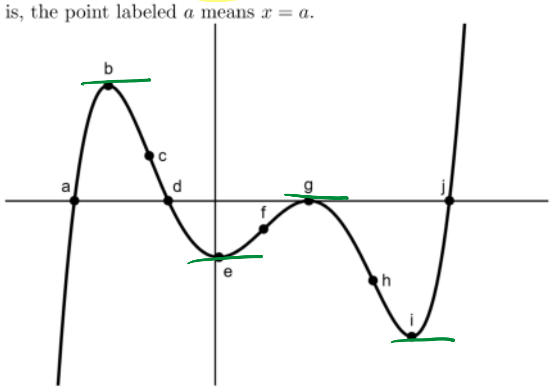


(c) Determine the location of any local extrema of $f(x)$.

Local Minimum: @ $x = a$, $x = j$

Local Maxima: @ $x = d$

7. Consider the graph of $f'(x)$ below. Assume each label corresponds with an $x-$value. That is, the point labeled a means $x = a$.



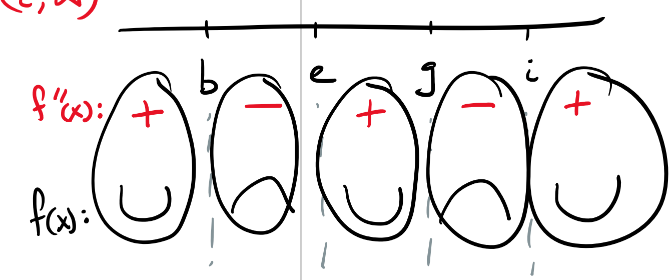
(a) Determine the critical numbers of $f'(x)$, that is, where $f''(x)$ DNE or $f''(x) = 0$.

$f''(x)=0$: $\boxed{x = b, e, g, i}$

(b) Determine the intervals where $f(x)$ is concave up/down.

$f(x)$ is Concave Up: $(-\infty, b), (e, g), (i, \infty)$

$f(x)$ is Concave Down: $(b, e), (g, i)$

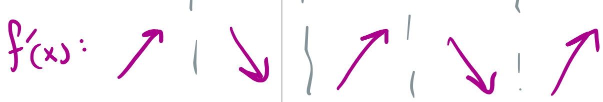


(c) Determine the location of any inflection points of $f(x)$.

$\boxed{@ \ x = b, \ e, \ g, \ i}$

(d) Determine the intervals where $f'(x)$ is increasing/decreasing.

$\Rightarrow f''(x) > 0 \Rightarrow f(x)$ is Concave Up



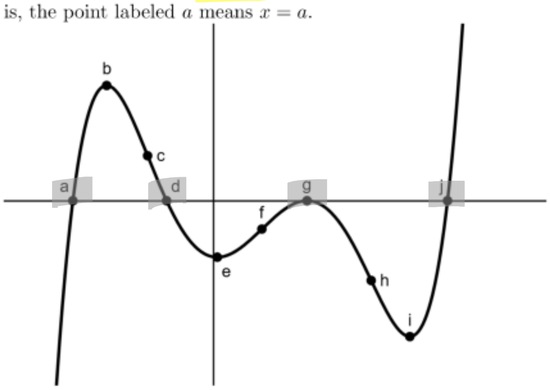
(e) Determine the location of any local extrema of $f'(x)$

Local Max of $f'(x)$: @ $x = b$, $x = g$

Local Min of $f'(x)$: @ $x = e$, $x = i$

8. Consider the graph of $f''(x)$ below. Assume each label corresponds with an $x-$value. That is, the point labeled a means $x = a$.

$f''(x) = 0$



(a) Determine the critical numbers of $f'(x)$, that is, where $f''(x)$ DNE or $f''(x) = 0$.

$f''(x) = 0$ $\quad$ $x = a, d, g, j$

(b) Determine the intervals where $f(x)$ is concave up/down.

(c) Determine the location of any inflection points of $f(x)$.

Inflection Points: @ $x = a$, $x = d$, $x = j$

		a		d		g		j	
$f''(x)$:	$-$		$+$		$-$		$-$		$+$
$f'(x)$:	$\searrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$
$f(x)$:	$\cap$		$\cup$		$\cap$		$\cap$		$\cup$

(d) Determine the intervals where $f'(x)$ is increasing/decreasing.

(e) Determine the location of any local extrema of $f'(x)$

Local min. of $f'(x)$: @ $x = a$, $x = j$

Local Max of $f'(x)$: @ $x = d$

9. Consider the function $f(x) = 5x^4 - \frac{40}{3}x^3$. $f'(x) = 20x^2(x-2)$

(a) Determine the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing, as well as the $x-$values of any local extrema.

$f'(x) = 20x^3 - 40x^2$

$f'(x)$ DNE : N/A

$f'(x) = 0$

$20x^3 - 40x^2 = 0$

$20x^2(x-2) = 0$

$x=0$
$x=2$

	(−1)	0	(1)	2	(10)
$f'(x)$:	−		−		+
$f(x)$:	↘		↘		↗

$f(x)$ Decreasing: $(-\infty, 0), (0, 2)$
$f(x)$ Increasing: $(2, \infty)$

Local Max: None

Local Min: @ $x = 2$

Typo

(b) Determine the intervals where $f(x)$ is concave up and where $f(x)$ is concave down, as well as the $x-$values of any ~~local extrema~~. Inflection Points

$f''(x) = 20x(3x-4)$

$f''(x) = 60x^2 - 80x$

$f''(x)$ DNE: N/A

$f''(x) = 0$

$60x^2 - 80x = 0$

$20x(3x-4) = 0$

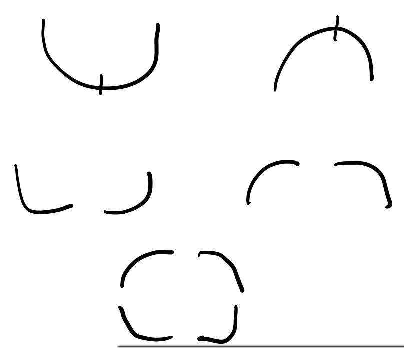
$x = 0$
$x = \frac{4}{3}$

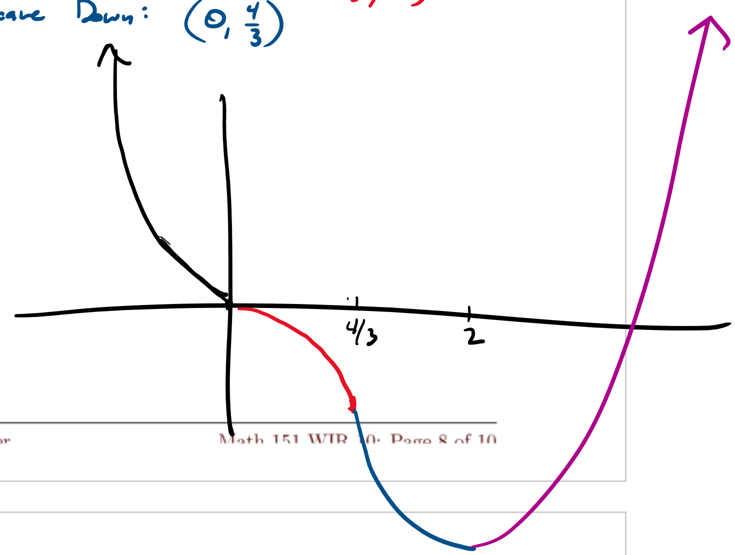
	(−1)	0	(1)	4/3	(10)
	(−)(−)		(+)(−)		(+)(+)
$f''(x)$:	+		−		+
$f(x)$:	∪		∩		∪

Concave Up: $(-\infty, 0)$, $(\frac{4}{3}, \infty)$
Concave Down: $(0, \frac{4}{3})$

Inflection Points @ $x = 0$, $x = \frac{4}{3}$

(c) Sketch the function.





TEXAS A&M UNIVERSITY
Math Learning Center

Math 151 - Fall 2024
Week-In-Review 10

10. Consider the function $f(x) = x^{1/3}(6-x)^{2/3}$.

$$f'(x) = \frac{-3x+6}{3x^{2/3}(6-x)^{1/3}} = \frac{-(x-2)}{x^{2/3}(6-x)^{1/3}}$$

Always +

(a) Determine the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing, as well as the $x-$values of any local extrema.

$$f'(x) = x^{1/3} \cdot \frac{2}{3}(6-x)^{-1/3}(-1) + (6-x)^{2/3} \cdot \frac{1}{3}x^{-2/3}$$

$$= \frac{x^{2/3}}{x^{2/3}} \cdot \frac{-2x^{1/3}}{3(6-x)^{1/3}} + \frac{(6-x)^{2/3}}{3x^{2/3}} \cdot \frac{(6-x)^{1/3}}{(6-x)^{1/3}}$$

$f'(x)$ DNE : $x=0$, $x=6$

$f'(x) = 0$

$-2x + (6-x) = 0$

$-3x + 6 = 0$

$-3x = -6$

$x = 2$

	(−1)	0	(1)	2	(3)	6	(10)
	$\frac{(-)(-)}{(+)}$		$\frac{(-)(-)}{(+)}$		$\frac{(-)(+)}{(+)}$		$\frac{(-)(+)}{(-)}$
$f(x)$:	+		+		−		+
$f(x)$:	↗		↗		↘		↗

$x^{2/3}$ $\frac{\quad}{3(6-x)^{1/3}} + \frac{\quad}{3x^{2/3}} \cdot (6-x)^{1/3}$

$-3x = -6$

$x = 2$

f(x): ↗ ↗ ↘ ↗

Local Max: @ x=2

Local Min: @ x=6

$f'(x)$ DNE : x=0, x=6

$$f'(x) = \frac{-2x + (6-x)}{3x^{2/3}(6-x)^{1/3}} = 0$$

Typo

(b) Determine the intervals where $f(x)$ is concave up and where $f(x)$ is concave down, as well as the $x-$values of any ~~local extrema~~. Inflection Points

Note: $f''(x) = \dfrac{-8}{(6-x)^{4/3}x^{5/3}} = 0$

Always +

$f''(x)$ DNE: x=0, x=6

$f''(x) = 0$ $\quad -8 = 0$ $\quad$ N/A

(−1) (1) (10)

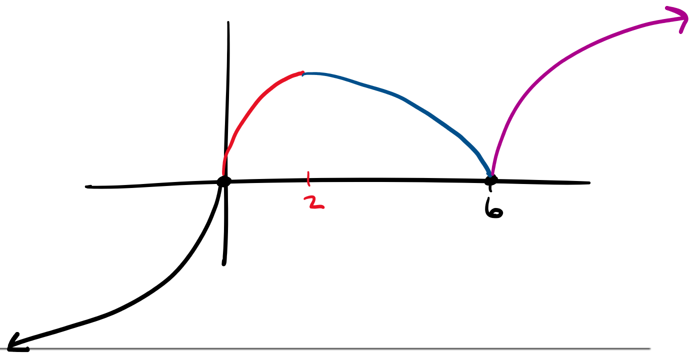
0 6

$f''(x)$: $\frac{(-)}{(-)}$ $\frac{(-)}{(+)}$ $\frac{(-)}{(+)}$

f(x): + ∪ — ∩ — ∩

Inflection Points: @ x=0

(c) Sketch the function. Note: $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$.

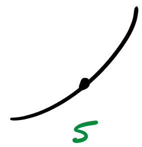


11. Let $f(t)$ be the temperature at time t where you live, and suppose that at time $t = 5$ you are uncomfortably hot. How would you feel about the given data in each case?

(a) $f'(5) = 2, f''(5) = 4$

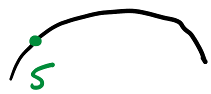
(5)
f'(x): +
f(x): ↗

(5)
f''(x): +
f(x): ∪



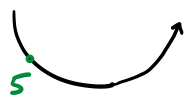
Very Bad

(b) $f'(5) = 2, f''(5) = -4$



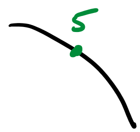
Bad but hopeful?

(c) $f'(5) = -2, f''(5) = 4$



Good but worried?

(d) $f'(5) = -2, f''(5) = -4$



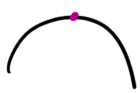
Oh Thank goodness

(e) $f'(5) = 0, f''(5) = 4$

This is the Second Derivative Test.

Horrible

(f) $f'(5) = 0, f''(5) = -4$



Local Max.

Great.