Trig Identities

$\sin^2 x + \cos^2 x = 1 \qquad \sin x \cos x = \frac{1}{2}\sin 2x \qquad \sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \sec^2 x = \tan^2 x + 1$

$\int \sin^n x \cos^m x\,dx \begin{cases} n \text{ is odd}, & u=\cos x,\ \sin^2 x = 1-\cos^2 x \\ m \text{ is odd}, & u=\sin x,\ \cos^2 x = 1-\sin^2 x \end{cases}$

Math 152/172 **WEEK in REVIEW 4** **Spring 2025.**

1. Integrate the following trigonometric functions.

(a) $\int \sin^3 x \cos^4 x\,dx$

$\sin^3 x = \sin x \cdot \sin^2 x = \sin x\,(1-\cos^2 x)$

$u = \cos x$
$du = -\sin x\,dx$

$= \int \sin x (1-\cos^2 x)\cos^4 x\,dx = -\int (1-u^2)u^4\,du$

$= -\int (u^4 - u^6)\,du = +\left(-\frac{u^5}{5} + \frac{u^7}{7}\right) + C$

$= \boxed{\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C}$

(b) $\int \sin^2(2x)\cos^3(2x)\,dx =$

$\cos^3(2x) = \cos(2x)\cos^2(2x) = \cos(2x)(1-\sin^2 2x)$

$= \int \sin^2(2x)\cos(2x)(1-\sin^2(2x))\,dx$

$u = \sin 2x$
$du = 2\cos 2x\,dx$
$\cos 2x\,dx = \frac{du}{2}$

$= \int u^2(1-u^2)\frac{du}{2} = \frac{1}{2}\int (u^2-u^4)\,du = \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$

$= \boxed{\frac{1}{2}\left(\frac{\sin^3(2x)}{3} - \frac{\sin^5(2x)}{5}\right) + C}$

(c) $\int \sin^5 x \cos^7 x\,dx$

$\sin^5 x = \sin x \cdot \sin^4 x = \sin x(\sin^2 x)^2 = \sin x(1-\cos^2 x)^2$

$\int \sin x(1-\cos^2 x)^2 \cos^7 x\,dx$

$u = \cos x$
$du = -\sin x\,dx$

$= -\int (1-u^2)^2 u^7\,du = -\int (1-2u^2+u^4)u^7\,du = -\int (u^7 - 2u^9 + u^{11})\,du$

$-\frac{u^8}{8} + 2\frac{u^{10}}{10} + \frac{u^{12}}{12} + C = \boxed{-\frac{\cos^8 x}{8} + \frac{1}{5}\cos^{10} x + \frac{\cos^{12} x}{12} + C}$

if you do $u = \sin x$, $du = \cos x\,dx$ and $\cos^7 x = \cos x \cdot \cos^6 x = \cos x(\cos^2 x)^3 = \cos x(1-\sin^2 x)^3$

$\int u^5(1-u^2)^3\,du = \int u^5(1-3u^2+3u^4-u^6)\,du$

$= \int (u^5 - 3u^7 + 3u^9 - u^{11})\,du$

$= \frac{u^6}{6} - \frac{3u^8}{8} + \frac{3u^{10}}{10} - \frac{u^{12}}{12} + C$

$= \boxed{\frac{\sin^6 x}{6} - \frac{3\sin^8 x}{8} + \frac{3\sin^{10} x}{10} - \frac{\sin^{12} x}{12} + C}$

(d) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\int (1-\cos 2x)\,dx = \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$

(e) $\int \cos^4 x\,dx = \int (\cos^2 x)^2 dx = \int \left[\frac{1+\cos 2x}{2}\right]^2 dx = \int \left(\frac{1}{4}(1+2\cos 2x+\cos^2 2x)\right) dx$

$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)dx = \frac{1}{4}\left(x + 2\cdot\frac{1}{2}\sin 2x\right) + \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx$

$= \frac{1}{4}(x+\sin 2x) + \frac{1}{8}\left(x + \frac{1}{4}\sin 4x\right) + C = \boxed{\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C}$

$\sin x\cos x = \frac{1}{2}\sin 2x$

(f) $\int \sin^4 x\cos^4 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^4 dx = \frac{1}{16}\int \sin^4 2x\,dx = \frac{1}{16}\int (\sin^2 2x)^2\,dx$ $\quad \sin^2 2x = \frac{1-\cos 4x}{2}$

$= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 dx = \frac{1}{16}\int \frac{1}{4}(1 - 2\cos 4x + \cos^2 4x)\,dx$

$= \frac{1}{64}\int\left(1 - 2\cos 4x + \frac{1+\cos 8x}{2}\right)dx = \frac{1}{64}\int\left(\frac{3}{2} - 2\cos 4x + \frac{1}{2}\cos 8x\right)dx$

$= \boxed{\frac{1}{64}\left(\frac{3}{2}x - \frac{2}{4}\sin 4x + \frac{1}{2}\cdot\frac{1}{8}\sin 8x\right) + C}$

(g) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx \quad \left| \begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array} \right| = -\int \frac{du}{u} = -\ln|u| + C = \boxed{-\ln|\cos x| + C}$

$= \boxed{\ln|\sec x| + C}$

$\tan^2 x + 1 = \sec^2 x$ or $\tan^2 x = \sec^2 x - 1$

(h) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int dx = \boxed{\tan x - x + C}$

(i) $\int \tan^3 x\,dx = \int \tan x(\tan^2 x)\,dx = \int \tan x(\sec^2 x - 1)\,dx$

$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx = \int u\,du + \ln|\cos x| + C$

$u = \tan x$
$du = \sec^2 x\,dx$

$= \frac{u^2}{2} + \ln|\cos x| + C$

$= \boxed{\frac{\tan^2 x}{2} + \ln|\cos x| + C}$

$\int \sec^m x \tan^n x\,dx = \begin{cases} m \text{ is even}, & u = \tan x,\ du = \sec^2 x\,dx,\ \sec^2 x = \tan^2 x + 1 \\ m \text{ is odd}, & u = \sec x,\ du = \sec x\tan x\,dx,\ \tan^2 x = \sec^2 x - 1 \end{cases}$

(j) $\int \sec^4 x\tan^3 x\,dx = \left| \begin{array}{l} \sec^4 x = \sec^2 x\cdot\sec^2 x \\ = \sec^2 x(\tan^2 x + 1) \end{array} \right| = \int \sec^2 x(\tan^2 x+1)\tan^3 x\,dx \quad \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x\,dx \end{array} \right|$

$= \int (u^2+1)u^3\,du = \int (u^5 + u^3)\,du = \frac{u^6}{6} + \frac{u^4}{4} + C = \boxed{\frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C}$

$$= \sec^2 x\,(\tan^2 x + 1)$$

$$= \int (u^2+1)u^3\,du = \int (u^5 + u^3)\,du = \frac{u^6}{6} + \frac{u^4}{4} + C = \boxed{\frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C}$$

(k) $\int \sec^3 x \tan^3 x\,dx = \int (\sec x \tan x)\sec^2 x \tan^2 x\,dx$ $\left|\begin{array}{l} u = \sec x \\ du = \sec x \tan x\,dx \\ \tan^2 x = \sec^2 x - 1 \end{array}\right|$

$$= \int u^2(u^2-1)\,du = \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

(l) $\int \sec x\,dx = \int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x}\,dx$ $\left|\begin{array}{l} u = \tan x + \sec x \\ du = (\sec^2 x + \sec x \tan x)\,dx \end{array}\right| = \int \frac{du}{u}$

$$= \ln|u| + C = \boxed{\ln|\tan x + \sec x| + C}$$

(m) $\int \sec^2 x\,dx = \tan x + C$

$$\int u v'\,dx = uv - \int u' v\,dx$$

by parts

(n) $\int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx$ $\left|\begin{array}{ll} u = \sec x & v' = \sec^2 x \\ u' = \sec x \tan x & v = \tan x \end{array}\right| = \sec x \tan x - \int \sec x \tan x \tan x\,dx$

$$= \sec x \tan x - \int \sec x \tan^2 x\,dx \;\underline{\underline{\tan^2 x = \sec^2 x - 1}}\; \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

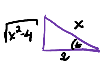
$$\int \sec^3 x\,dx = \boxed{\frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C}$$

$\sqrt{a^2 - x^2} \longrightarrow u = a\sin t$ $\quad\sqrt{x^2 - a^2} \longrightarrow u = a\sec t$

$\sqrt{a^2 + x^2} \longrightarrow u = a\tan t$

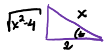
2. Do an appropriate substitution and evaluate the integral.

(a) $\int \frac{\sqrt{x^2-4}}{x^4}\,dx$ $\left|\begin{array}{l} x = 2\sec t \\ \sqrt{x^2-4} = \sqrt{4\sec^2 t - 4} = \sqrt{4\underbrace{(\sec^2 t - 1)}_{\tan^2 t}} \\ \sqrt{x^2-4} = \sqrt{4\tan^2 t} = 2\tan t \\ dx = 2\sec t \tan t\,dt \end{array}\right| = \int \frac{\overset{\sqrt{x^2-4}}{2\tan t}}{\underset{x^4}{(2\sec t)^4}}\,\underbrace{2\sec t \tan t\,dt}_{dx}$



$x = 2\sec t$

$$= \int \frac{4\tan^2 t \sec t}{16\sec^4 t}\,dt = \int \frac{\tan^2 t}{4\sec^3 t}\,dt = \frac{1}{4}\int \cos^3 t \cdot \frac{\sin^2 t}{\cos^2 t}\,dt$$



$x = 2\sec t$

$\sec t = \frac{x}{2}$

$\cos t = \frac{2}{x}$

$\sin t = \frac{\sqrt{x^2-4}}{x}$

$dx = 2\sec t\tan t\, dt$

$$= \int \frac{4\tan^2 t\sec t}{16\sec^4 t}\,dt = \int \frac{\tan^2 t}{4\sec^3 t}\,dt = \frac{1}{4}\int \cos^3 t \cdot \frac{\sin^2 t}{\cos^2 t}\,dt$$

$$= \frac{1}{4}\int \cos t\,\sin^2 t\,dt = \left| \begin{array}{l} u = \sin t \\ du = \cos t\,dt \end{array} \right| = \frac{1}{4}\int u^2\,du = \frac{1}{4}\,\frac{u^3}{3} + C$$

$$= \frac{1}{12}(\sin t)^3 + C = \boxed{\frac{1}{12}\left(\frac{\sqrt{x^2-4}}{x}\right)^3 + C}$$

(b) $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{x\cdot x^2\,dx}{\sqrt{x^2+4}}$ $\left| \begin{array}{l} u = x^2+4 \to x^2 = u-4 \\ du = 2x\,dx \to x\,dx = \frac{du}{2} \end{array} \right|$ $= \displaystyle\int \frac{u-4}{\sqrt{u}}\,\frac{du}{2} = \frac{1}{2}\int\left(u^{1/2} - 4u^{-1/2}\right)du$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2} - 4\,\frac{u^{1/2}}{1/2}\right) + C = \boxed{\frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C}$$

$\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

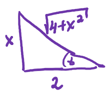
$x = 2\tan t$

$dx = 2\sec^2 t\,dt$

$\sqrt{x^2+4} = \sqrt{4\tan^2 t + 4} = \sqrt{4\underbrace{(\tan^2 t + 1)}_{\sec^2 t}} = 2\sec t$

$x = 2\tan t$

$\tan t = \frac{x}{2}$



$\sec t = \frac{1}{\cos t}$

$= \frac{\sqrt{4+x^2}}{2}$

$$= \int \frac{\overbrace{8\tan^3 t}^{x^3}\,\overbrace{(\cancel{2}\sec^{\cancel{2}} t)\,dt}^{dx}}{\underbrace{\cancel{2}\,\cancel{\sec t}}_{\sqrt{x^2+4}}} = 8\int \tan^3 t\sec t\,dt \quad \left| \begin{array}{l} u = \sec t \\ du = \sec t\tan t\,dt \\ \tan^2 t = \sec^2 t - 1 = u^2 - 1 \end{array} \right| = 8\int (u^2-1)\,du$$

$$= 8\left(\frac{u^3}{3} - u\right) + C = 8\left(\frac{\sec^3 t}{3} - \sec t\right) + C = 8\left(\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \frac{\sqrt{x^2+4}}{2}\right) + C$$

$$= \boxed{\frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C}$$

(c) $\displaystyle\int \frac{x^2\,dx}{\sqrt{16-x^2}}$

$\boxed{x = 4\sin t} \to \sin t = \frac{x}{4}$, $\boxed{t = \arcsin\frac{x}{4}}$

$dx = 4\cos t\,dt$

$\sqrt{16-x^2} = \sqrt{16-16\sin^2 t} = \sqrt{16(1-\sin^2 t)} = 4\cos t$

$\sqrt{16-x^2} = 4\cos t$

$\boxed{\cos t = \frac{\sqrt{16-x^2}}{4}}$

$\sin 2t = 2\sin t\cos t$

$$= \int \frac{16\sin^2 t\;4\cos t\,dt}{4\cos t} = 16\int \sin^2 t\,dt = 16\int \frac{1-\cos 2t}{2}\,dt = 8\left(t - \frac{1}{2}\sin 2t\right) + C$$

$$= 8\left(\arcsin\frac{x}{4} - \frac{1}{\cancel{2}}\cdot\cancel{2}\sin t\cos t\right) + C$$

$$= \boxed{8\left(\arcsin\frac{x}{4} - \frac{x}{4}\,\frac{\sqrt{16-x^2}}{4}\right) + C}$$

(d) $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-25x^2}} = \int \frac{x^2\,dx}{\sqrt{25\left(\frac{9}{25}-x^2\right)}} = \frac{1}{5}\int \frac{x^2\,dx}{\sqrt{\frac{9}{25}-x^2}}$

$x = \frac{3}{5}\sin t \to \sin t = \frac{5x}{3}$, $t = \arcsin\left(\frac{5x}{3}\right)$

$dx = \frac{3}{5}\cos t\,dt$

$\sqrt{\frac{9}{25}-x^2} = \sqrt{\frac{9}{25} - \frac{9}{25}\sin^2 t} = \sqrt{\frac{9}{25}\cos^2 t} = \frac{3}{5}\cos t$

$\frac{3}{5}\cos t = \sqrt{\frac{9}{25}-x^2}$

$\cos t = \frac{5}{3}\sqrt{\frac{9}{25}-x^2}$

$= \sqrt{\frac{25}{9}\left(\frac{9}{25}-x^2\right)}$

$= \left(1 - \frac{25}{9}x^2\right)^{1/2}$

$$= \frac{1}{5}\int \frac{\frac{9}{25}\sin^2 t\;\cancel{\frac{3}{5}\cos t}\,dt}{\cancel{\frac{3}{5}\cos t}} = \frac{9}{125}\int \sin^2 t\,dt = \frac{9}{125}\int \frac{1-\cos 2t}{2}\,dt$$

$\sin 2t = 2\sin t\cos t$

$$= \frac{9}{250}\left(t - \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{9}{250}\left(t - \sin t\cos t\right) + C = \boxed{\frac{9}{250}\left(\arcsin\frac{5x}{3} - \frac{5x}{3}\left(1 - \frac{25}{9}x^2\right)^{1/2}\right) + C}$$

$$= \frac{9}{250}\left(t - \sin t\cos t\right) + C = \boxed{\frac{9}{250}\left(\arcsin \frac{5x}{3} - \frac{5x}{3}\left(1 - \frac{25}{9}x^2\right)^{1/2}\right) + C}$$

(e) $\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{dx}{\sqrt{4\left(x^2+\frac{9}{4}\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+\frac{9}{4}}}$

$x = \frac{3}{2}\tan t \rightarrow \tan t = \frac{2x}{3}$

$dx = \frac{3}{2}\sec^2 t\, dt$

$\sqrt{x^2+\frac{9}{4}} = \sqrt{\frac{9}{4}\tan^2 t + \frac{9}{4}} = \sqrt{\frac{9}{4}\sec^2 t} = \frac{3}{2}\sec t$

$\sec t = \frac{2}{3}\sqrt{x^2+\frac{9}{4}} = \sqrt{\frac{4}{9}x^2+1}$

$$= \frac{1}{2}\int \frac{\cancel{\frac{3}{2}}\sec^{\cancel{2}} t\, dt}{\cancel{\frac{3}{2}}\cancel{\sec t}} = \frac{1}{2}\int \sec t\, dt$$

$$= \frac{1}{2}\ln\left|\sec t + \tan t\right| + C$$

$$= \boxed{\frac{1}{2}\ln\left|\sqrt{\frac{4}{9}x^2+1} + \frac{2x}{3}\right| + C}$$

(f) $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}}$

[complete the square] $x^2+6x+13 = (x^2+6x+9)-9+13 = (x+3)^2+4$

$$\int \frac{dx}{\sqrt{(x+3)^2+4}}$$

$x+3 = 2\tan t$ or $x = 2\tan t - 3$

$dx = 2\sec^2 t\, dt$

$\sqrt{(x+3)^2+4} = \sqrt{4\tan^2 t+4} = \sqrt{4\sec^2 t} = 2\sec t$

$x+3 = 2\tan t \rightarrow \tan t = \frac{x+3}{2}$

$2\sec t = \sqrt{(x+3)^2+4} \rightarrow \sec t = \frac{\sqrt{(x+3)^2+4}}{2}$

$$= \int \frac{2\sec^2 t\, dt}{2\sec t} = \int \sec t\, dt = \ln|\sec t + \tan t| + C$$

$$= \boxed{\ln\left|\frac{x+3}{2} + \frac{\sqrt{(x+3)^2+4}}{2}\right| + C}$$

(g) $\displaystyle\int x^2\sqrt{3+2x-x^2}\,dx$

$3+2x-x^2 = 3-(x^2-2x)$

$= 3-(x^2-2x+1)+1 = 4-(x-1)^2$

$$= \int x^2\sqrt{4-(x-1)^2}\,dx$$

$x-1 = 2\sin t \longrightarrow x = 2\sin t+1$

$dx = 2\cos t\, dt$

$\sqrt{4-(x-1)^2} = \sqrt{4-4\sin^2 t} = \sqrt{4\cos^2 t} = 2\cos t$

$$= \int (2\sin t+1)^2\, 2\cos t\, 2\cos t\, dt = 4\int \cos^2 t\,(2\sin t+1)^2\, dt$$

$$= 4\int \cos^2 t\,(4\sin^2 t + 4\sin t + 1)\, dt$$

$$= 16\int \cos^2 t\sin^2 t\, dt + 16\int \cos^2 t\sin t\, dt + 4\int \cos^2 t\, dt$$

$\cos^2 t\sin^2 t = \left(\frac{1}{2}\sin 2t\right)^2$

$= \frac{1}{4}\sin^2 2t = \frac{1}{4}\left(\frac{1-\cos 4t}{2}\right)$

$= \frac{1}{8}(1-\cos 4t)$

$u = \cos t$

$du = \sin t\, dt$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$$= \frac{16}{8}\int (1-\cos 4t)\,dt + 16\int u^2\,du + \frac{4}{2}\int (1+\cos 2t)\,dt$$

$$= 2\left(t - \frac{1}{4}\sin 4t\right) + 16\frac{u^3}{3} + 2\left(t+\frac{1}{2}\sin 2t\right) + C$$

$$= 2\left(t-\frac{1}{4}\sin 4t\right) + \frac{16}{3}\cos^3 t + 2\left(t+\frac{1}{2}\sin 2t\right) + C$$

$$= 4t - \frac{1}{2}\sin 4t + \sin 2t + \frac{16}{3}\cos^3 t + C$$

$\sin 2t = 2\sin t\cos t$

$\sin 4t = \sin 2(2t) = 2\sin 2t\cos 2t$

$= 2(2\sin t\cos t)(\underbrace{\cos^2 t - \sin^2 t}_{\cos 2t})$

$\boxed{\sin 4t = 4\sin t\cos^3 t - 4\sin^3 t\cos t}$

$4t - \dots 4(\sin t\cos^3 t - \sin^3 t\cos t) + 2\sin t\cos t + \frac{16}{3}\cos^3 t + C$

$$= 4t - \frac{4}{2}\left(\sin t \cos^3 t - \sin^3 t \cos t\right) + 2 \sin t \cos t + \frac{16}{3} \cos^3 t + C$$

$$x-1 = 2\sin t \longrightarrow \sin t = \frac{x-1}{2} \longrightarrow t = \arcsin \frac{x-1}{2}$$

$$2\cos t = \sqrt{4-(x-1)^2} \longrightarrow \cos t = \frac{\sqrt{4-(x-1)^2}}{2}$$

$$= \boxed{4 \arcsin \frac{x-1}{2} - 2\left(\frac{x-1}{2} \frac{(4-(x-1)^2)^{3/2}}{8} - \frac{(x-1)^3}{8} \cdot \frac{(4-(x-1)^2)^{1/2}}{2}\right) + 2 \frac{x-1}{2} \frac{(4-(x-1)^2)^{1/2}}{2} + \frac{16}{3} \frac{(4-(x-1)^2)^{3/2}}{8} + C}$$