1. Evaluate the definite integral $\int_0^{\pi/4} \sec^2 x e^{\tan x} dx$.

$u = \tan x$
$du = \sec^2 x\, dx$
$u(0) = \tan 0 = 0$
$u(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

$= \int_0^1 e^u du$
$= e^u \Big|_0^1 = \boxed{e-1}$

2. Evaluate the definite integral $\int_0^1 \frac{x\,dx}{\sqrt{1+x^2}}$.

$u = 1+x^2$
$du = 2x dx \rightarrow x dx = \frac{du}{2}$
$u(0) = 1+0^2 = 1$
$u(1) = 1+1^2 = 2$

$= \int_1^2 \frac{\frac{du}{2}}{\sqrt{u}} = \frac{1}{2}\int_1^2 u^{-1/2} du = \cancel{\frac{1}{2}} \frac{u^{1/2}}{\cancel{1/2}}\Big|_1^2 = \boxed{\sqrt{2}-1}$

3. Evaluate the indefinite integral $\int x^5\sqrt{x^3+1}\,dx$

$u = x^3+1 \rightarrow x^3 = u-1$
$du = 3x^2 dx \rightarrow x^2 dx = \frac{du}{3}$

$= \int x^3 \cdot x^2 \sqrt{x^3+1}\, dx$

$= \int (u-1)\sqrt{u}\,\frac{du}{3} = \frac{1}{3}\int \left(u^{3/2} - u^{1/2}\right) du = \frac{1}{3}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C$

$= \boxed{\frac{1}{3}\left(\frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2}\right) + C}$

4. Evaluate the definite integral $\int_0^{\pi/8} \underbrace{\sin^3(2x)}_{u^3}\underbrace{\cos(2x)\,dx}_{du/2}$

$u = \sin 2x$
$du = 2\cos 2x\, dx$
$x = 0 \rightarrow u = \sin 0 = 0$
$x = \frac{\pi}{8} \rightarrow u = \sin\frac{2\pi}{8} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$= \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}} u^3 du = \frac{u^4}{8}\Big|_0^{\frac{\sqrt{2}}{2}} = \frac{1}{8}\cdot\frac{1}{4} = \boxed{\frac{1}{32}}$

5. Evaluate the indefinite integral $\int \frac{\sec\theta\ \tan\theta\, d\theta}{4+\sec\theta}$.

$\int u v' dx = uv - \int u' v \, dx$

1) polynomial times $\begin{bmatrix} \exp \\ \sin \\ \cos \end{bmatrix} \longrightarrow$ $u =$ polynomial, $v' = \begin{bmatrix} \exp \\ \sin \\ \cos \end{bmatrix}$

2) ln, arcsin, arctan $\longrightarrow$ $u = \begin{matrix} \ln \\ \arcsin \end{matrix} \longrightarrow v' =$ whatever is left.

---

6. Evaluate the indefinite integral $\int x^3 \ln x \, dx$.

$u = \ln x$, $v' = x^3$

$u' = \frac{1}{x}$, $v = \frac{x^4}{4}$

$$= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx = \boxed{\frac{x^4}{4} \ln x - \frac{x^4}{16} + C}$$

7. Evaluate the definite integral $\int_1^{\sqrt{3}} \underbrace{1}_{v'} \cdot \underbrace{\arctan\left(\frac{1}{x}\right)}_{u} dx$.

$u = \arctan\left(\frac{1}{x}\right)$, $v' = 1$

$u' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)$, $v = x$

$= -\frac{1}{x^2\left[1+\frac{1}{x^2}\right]}$

$u' = -\frac{1}{x^2+1}$

$$= x \arctan\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_{1=x}^{\sqrt{3}=x} x \left(+\frac{1}{x^2+1}\right) dx$$

$z = x^2+1$

$dz = 2x\,dx \longrightarrow x\,dx = \frac{dz}{2}$

$x = 1 \longrightarrow z = 1^2+1 = 2$

$x = \sqrt{3} \longrightarrow z = (\sqrt{3})^2+1 = 4.$

$$= x \arctan\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \frac{1}{2}\int_{z=2}^{z=4} \frac{dz}{z}$$

$$= \sqrt{3} \arctan \frac{1}{\sqrt{3}} - \arctan 1 + \frac{1}{2} \ln|z| \Big|_2^4 = \boxed{\sqrt{3} \cdot \frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2)}$$

($\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$, $\arctan 1 = \frac{\pi}{4}$)

8. Evaluate the indefinite integral $\int (x^3 + x^2 + x + 1) e^x dx$.

$$= \boxed{(x^3+x^2+x+1)e^x - (3x^2+2x+1)e^x + (6x+2)e^x - 6e^x + C}$$

	D(u)	I(v')
+	x^3+x^2+x+1	e^x
−	$3x^2+2x+1$	e^x
+	$6x+2$	e^x
−	6	e^x
	0	e^x

9. Evaluate the definite integral $\int_0^{\pi} \sin(2x) e^{\cos x} dx$.

$\sin 2x = 2 \sin x \cos x$

$$= 2\int_{0=x}^{\pi=x} \sin x \cos x \, e^{\cos x} dx$$

$z = \cos x$

$dz = -\sin x \, dx$

$x = 0 \rightarrow z = \cos 0 = 1$

$x = \pi \rightarrow z = \cos \pi = -1$

$$= -2 \int_1^{-1} z e^z dz = 2 \int_{-1}^{1} z e^z dz$$

$$= 2\left[z e^z - e^z\right]^1$$

	D	I
+	z	e^z
−	1	e^z

$$= 2\left[z e^{z} - e^{z} \right]_{-1}^{1}$$

$$= 2\left(e - e - (-1)e^{-1} + e^{-1} \right) = \boxed{4e^{-1}}$$

z $+$	e^{z}
1 $-$	e^{z}
0	e^{z}

10. Evaluate the indefinite integral $\int e^{3x} \sin(2x)\, dx$.

$\left| \begin{array}{ll} u = \sin 2x & v' = e^{3x} \\ u' = 2\cos 2x & v = \frac{1}{3} e^{3x} \end{array} \right|$

$= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{3} \int \cos 2x\, e^{3x}\, dx$ $\left| \begin{array}{ll} u = \cos 2x & v' = e^{3x} \\ u' = -2\sin 2x & v = \frac{1}{3} e^{3x} \end{array} \right|$

$= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{3} \left[\frac{1}{3} e^{3x} \cos 2x - \int (-2\sin 2x) \frac{1}{3} e^{3x}\, dx \right]$

$= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{9} e^{3x} \cos 2x - \frac{4}{9} \int \sin(2x)\, e^{3x}\, dx$

D	I
$\sin 2x$ $+$	e^{3x}
$2\cos 2x$ $-$	$\frac{1}{3} e^{3x}$
$-4\sin 2x$ $+$	$\frac{1}{9} e^{3x}$
	$\frac{1}{27} e^{3x}$

the product goes inside the integral.

$\int e^{3x} \sin 2x\, dx = \frac{1}{3} e^{3x} \sin 2x - \frac{2}{9} e^{3x} \cos 2x - \frac{4}{9} \int e^{3x} \sin 2x\, dx$

denote $\int e^{3x} \sin 2x\, dx = I$

$I = \frac{1}{3} e^{3x} \sin 2x - \frac{2}{9} e^{3x} \cos 2x - \frac{4}{9} I$

solve it for I.

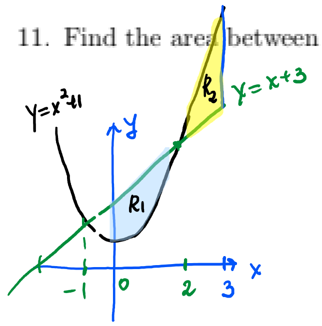
$I + \frac{4}{9} I = \frac{1}{3} e^{3x} \sin 2x - \frac{2}{9} e^{3x} \cos 2x$

$\frac{9}{13} \cdot \frac{13}{9} I = \left(\frac{1}{3} e^{3x} \sin 2x - \frac{2}{9} e^{3x} \cos 2x \right) \frac{9}{13}$

$I = \int e^{3x} \sin 2x\, dx = \frac{9}{13} \left(\frac{3}{\cancel{3}} e^{3x} \sin 2x - \frac{2}{\cancel{9}} e^{3x} \cos 2x \right) + C$

$= \frac{3}{13} e^{3x} \sin 2x - \frac{2}{13} e^{3x} \cos 2x + C$

11. Find the area between the curves $y = x^2 + 1$ and $y = x + 3$ when $0 \le x \le 3$.



points of intersection

$x^2+1 = x+3 \longrightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

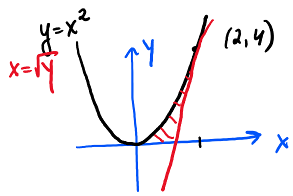
$x_1 = 2, \quad x_2 = -1$

R_1: $0 \le x \le 2$, $x^2+1 \le y \le x+3$

R_2: $2 \le x \le 3$, $x+3 \le y \le x^2+1$

$$A(R) = A(R_1) + A(R_2) = \int_0^2 \left([x+3] - [x^2+1]\right) dx + \int_2^3 \left((x^2+1) - (x+3)\right) dx = \dots$$

11.1 Find the area between the parabola $y = x^2$, the tangent line to the parabola at the point $(2,4)$, and above the x-axis.



integrate for y.

tangent line:

$y - 4 = y'(2)(x-2)$

$y'(x) = 2x, \quad y'(2) = 4.$

$y - 4 = 4(x-2)$

$\frac{y-4}{4} = x - 2 \longrightarrow x = 2 + \frac{y-4}{4}$

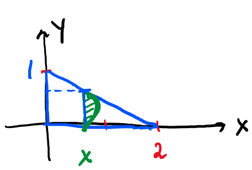
$x = 2 + \frac{y}{4} - 1$

$\boxed{x = 1 - \frac{y}{4}}$ tangent line

$0 \le y \le 4, \quad \sqrt{y} \le x \le 1 - \frac{y}{4}$

$$A = \int_0^4 \left(1 - \frac{y}{4} - \sqrt{y}\right) dy = \dots$$

12. Find the volume of the solid S whose base is the triangular region with vertices (0,0), (2,0), (0,1), and cross sections perpendicular to the x-axis are semicircles. $\longrightarrow$ integrate for x.



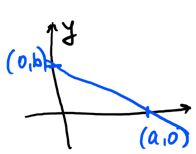
$\frac{x}{2} + y = 1 \longrightarrow x = 2y - 2 \longrightarrow \boxed{y = 1 - \frac{x}{2}}$

$0 \le x \le 2.$

$A(x) = A(\text{cross-section}) = \frac{1}{2}\pi r^2 \longrightarrow r = \frac{y}{2} = \frac{1}{2}\left(1 - \frac{x}{2}\right)$

$$V = \int_0^2 A\, dx = \frac{1}{2}\pi \int_0^2 \left[\frac{1}{2}\left(1 - \frac{x}{2}\right)\right]^2 dx$$

$$= \frac{\pi}{8} \int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \dots$$

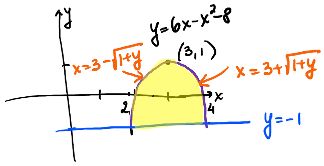


$\frac{x}{a} + \frac{y}{b} = 1$

13. Find the volume of the solid generated by rotating a plane region bounded by $y = 6x - x^2 - 8$ and the line $y = -1$ about the indicated line.

a) the y-axis b) $x = 1$ c) $y = 2$

d) $x = -2$ e) $y = -4$



$y = 6x - x^2 - 8$

$x^2 - 6x + 8 - y = 0.$

$x_1 = \dfrac{6 + \sqrt{36 - 4(8-y)}}{2} = \dfrac{6+\sqrt{4+4y}}{2} = \dfrac{6+2\sqrt{1+y}}{2}$

$x_1 = 3 + \sqrt{1+y}$

points of intersection

$y=-1$ and $y = 6x - x^2 - 8$

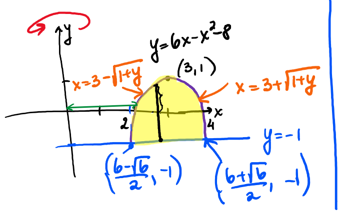
$-1 = 6x - x^2 - 8$

$x^2 - 6x + 7 = 0$

$x_1 = \dfrac{6 + \sqrt{36-28}}{2} = \dfrac{6+\sqrt{6}}{2}.$

$x_2 = \dfrac{6-\sqrt{6}}{2}.$

(a) about y-axis.



shells

dx

$\dfrac{6-\sqrt{6}}{2} \le x \le \dfrac{6+\sqrt{6}}{2}$

height $= 6x - x^2 - 8 + 1$

$= 6x - x^2 - 7$

radius $= x$

$V_y = 2\pi \displaystyle\int_{\frac{6-\sqrt{6}}{2}}^{\frac{6+\sqrt{6}}{2}} x(6x - x^2 - 7)\,dx$

washers.

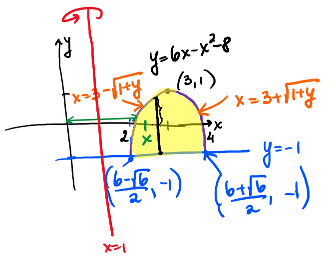
$dy.$

$r_{in} = 3 - \sqrt{1+y}$

$r_{out} = 3 + \sqrt{1+y}$

$V_y = \pi \displaystyle\int_{-1}^{1} \left[(3+\sqrt{1+y})^2 - (3-\sqrt{1+y})^2\right] dy$

(b) $x=1$



shells $\to dx$

$\dfrac{6-\sqrt{6}}{2} \le x \le \dfrac{6+\sqrt{6}}{2}$

height $= 6x - x^2 - 7$

radius $= x - 1$

$V_{x=1} = 2\pi \displaystyle\int_{\frac{6-\sqrt{6}}{2}}^{\frac{6+\sqrt{6}}{2}} (x-1)(6x - x^2 - 7)\,dx$

washers $\to dy$

$-1 \le y \le 1$

$r_{in} = 3 - \sqrt{1+y} - 1$

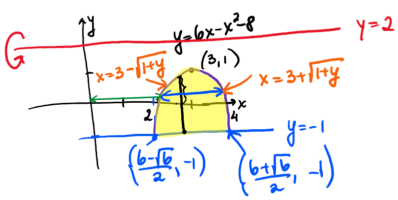
$= 2 - \sqrt{1+y}$

$r_{out} = 3 + \sqrt{1+y} - 1$

$= 2 + \sqrt{1+y}$

$V_{x=1} = \pi \displaystyle\int_{-1}^{1} \left[(2+\sqrt{1+y})^2 - (2-\sqrt{1+y})^2\right] dy$

c) $y=2.$



shells $\to dy$

$-1 \le y \le 1.$

height $= 3 + \sqrt{1+y} - (3 - \sqrt{1+y}) = 2\sqrt{1+y}$

$r = 2 - y,$

$V_{y=2} = 4\pi \displaystyle\int_{-1}^{1} (2-y)\sqrt{1+y}\,dy$

washers $\to dx$

$\dfrac{6-\sqrt{6}}{2} \le x \le \dfrac{6+\sqrt{6}}{2}$

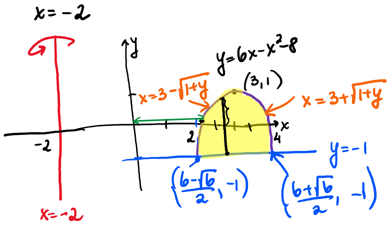
$r_{in} = 2 - (6x - x^2 - 8)$

$= 10 - 6x + x^2$

$r_{out} = 3$

$V_{y=2} = \pi \displaystyle\int_{\frac{6-\sqrt{6}}{2}}^{\frac{6+\sqrt{6}}{2}} \left[9 - (10 - 6x + x^2)^2\right] dx$

d) $x = -2$



shells $\to dx$

$\dfrac{6-\sqrt{6}}{2} \le x \le \dfrac{6+\sqrt{6}}{2}$

height $= 6x - x^2 - 7$

radius $= x + 2$

$V_{x=-2} = 2\pi \displaystyle\int_{\frac{6-\sqrt{6}}{2}}^{\frac{6+\sqrt{6}}{2}} (x+2)(6x - x^2 - 7)\,dx$

washers $\to dy$

$-1 \le y \le 1$

$r_{in} = 2 + (3 - \sqrt{1+y})$

$= 5 - \sqrt{1+y}$

$r_{out} = 2 + (3 + \sqrt{1+y})$

$= 5 + \sqrt{1+y}$

$V_{x=-2} = \pi \displaystyle\int_{-1}^{1} \left((5+\sqrt{1+y})^2 - (5-\sqrt{1+y})^2\right) dy$

e) $y=-4$

shells $\to dy$

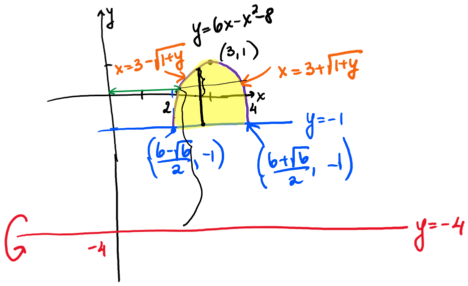
$-1 \le y \le 1$

height $= 2\sqrt{1+y}$

washers $\to dx$

$\dfrac{6-\sqrt{6}}{2} \le x \le \dfrac{6+\sqrt{6}}{2}$

e) $y=-4$



shells $\longrightarrow dy$

$-1 \le y \le 1$

height $= 2\sqrt{1+y}$

$r = 4+y$

$$V_{y=-4} = 4\pi \int_{-1}^{1} (4+y)\sqrt{1+y}\,dy$$

washers $\longrightarrow dx$

$$\frac{6-\sqrt{6}}{2} \le x \le \frac{6+\sqrt{6}}{2}$$

$r_{in} = 3$

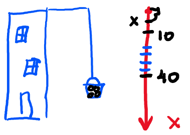
$r_{out} = 4 + (6x - x^2 - 8)$

$= 6x - x^2 - 4.$

$$V_{y=-4} = \pi \int_{\frac{6-\sqrt{6}}{2}}^{\frac{6+\sqrt{6}}{2}} \left[(6x - x^2 - 4)^2 - 9 \right] dx$$

14. A cable 40 feet long weighing 6 pounds per foot is hanging off the side of a ~~50~~ foot tall building. At the bottom of the cable is a bucket of rocks weighing 100 pounds. How much work is required to pull 10 feet of the cable to the top of the building?

ignore



$0 \le x \le 40$

~~density~~/weight of the cable. 6 lb/ft.

distance traveled.

$0 \le x \le 10. \longrightarrow dist = x$

$10 \le x \le 40 \longrightarrow dist = 10.$

$$W = \int_0^{10} 6x\,dx + \int_{10}^{40} 10 \cdot 6\,dx + \underbrace{(100)(10)}_{\text{the bucket.}} = \dots$$

15. A spring has a natural length of 20 cm. If a 10 J work is required to keep it stretched to a length 25 cm, how much work is done in stretching the spring from 30 cm to 80 cm?

$f(x) = kx$, k is a spring constant.

$20cm \to 0$

$25\,cm \to 25 - 20 = 5\,cm = 0.05\ (m)$

$30\,cm \to 30 - 20 = 10\,cm = 0.1\ (m)$

$80\,cm \to 80 - 20 = 60\,cm \to 0.6\ (m).$

Determine k: $10 = \int_0^{0.05} kx\,dx = k \cdot \frac{x^2}{2}\Big|_0^{\frac{1}{20}} = \frac{k}{2} \cdot \frac{1}{400} = \frac{k}{800}$

$10 = \frac{k}{800} \longrightarrow k = 8000$

$f(x) = 8000x$

$$W = \int_{0.1}^{0.6} 8000x\,dx = \dots$$

16. A spring has a natural length of 20 cm. If a force on 12 N is required to hold the spring stretched to a length of 40 cm, find the work required to stretch the spring from 30 cm to 70 cm.

$20\,cm \to 0$

$40\,cm \to 20\,cm \to 0.2.$

$f(x) = kx \longrightarrow 12 = 0.2k \longrightarrow k = 60$

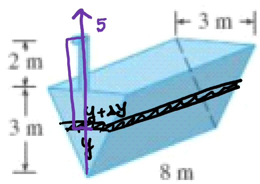
$f(x) = 60x$

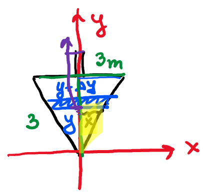
$30\,cm \to 10\,cm \to 0.1$

$70\,cm \to 50\,cm \to 0.5$

$$W = \int_{0.1}^{0.5} 60x\,dx = \dots$$

17. An 8 meter long tank in the shape of a triangular trough is full of water. Its vertical cross sections are isosceles triangles with base equal to its height of 3 meters. There is a 2 meter spout at the top of the tank. Set up the integral to find the work required to pump out the top 1.5 meters of water from the tank.





$W = \rho g \int_a^b (\text{volume})(\text{distance traveled})\, dy.$

weigh of water $= \rho g(\text{volume of the slice})$.

volume of the slice is $V = (2x)(8)\Delta y$

express x in terms of y.

similar triangles: $\frac{x}{\frac{3}{2}} = \frac{y}{3} \longrightarrow x = y/2$

$V = 2\left(\frac{y}{2}\right) \cdot 8\Delta y \longrightarrow 8y\Delta y.$

distance traveled by the slice. dist $= 5-y$

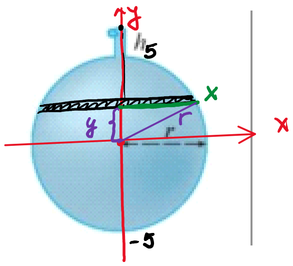
$0 \le y \le 1.5$

$$W = \rho g \int_{1.5}^{3} (5-y)\, 8y\, dy = 8\rho g \int_{1.5}^{3} y(5-y)\, dy.$$

$\rho = 10^3$ kg/m^3

$g = 9.81$ m/sec^2

18. A spherical tank with a radius r of 5 meters is completely full of water. The tank has a 0.5 meter spout h at the top.



integrate for y. , $-5 \le y \le 5$.

a slice of water is a cylinder of the height Δy

and the radius x

volume is $V = \pi x^2 \Delta y$

$x^2 = r^2 - y^2 \rightarrow V = \pi(r^2 - y^2)\Delta y = \pi(25 - y^2)\Delta y$

dist $= 0.5 + 5 - y = 5.5 - y$

(a) Set up an integral to find the work required to empty the full tank of water.

$$W = \rho g \int_{-5}^{5} \pi(25 - y^2)(5.5 - y)\,dy$$

(b) Set up an integral to find the work required to empty only half the tank of water.

$$W = \pi \rho g \int_{0}^{5} (25 - y^2)(5.5 - y)\,dy$$

(c) If you initially started out with only half a tank of water, set up an integral to find the work required to empty the tank.

$-5 \le y \le 0$

$$W = \pi \rho g \int_{-5}^{0} (25 - y^2)(5.5 - y)\,dy.$$