MASS AND SPRING SYSTEMS

Review

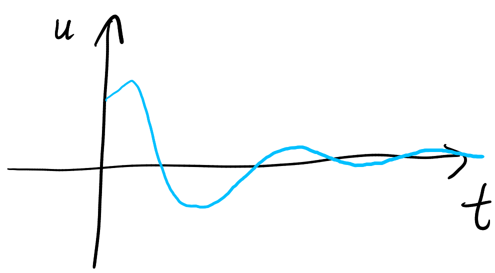
- Standard equation for **mass and spring system**

$$mu'' + \gamma u' + ku = F(t)$$

- $mg = kL$
- $\gamma = \dfrac{|F_{\text{damping}}|}{\text{speed}}$
- Standard equation for **electronic circuits**

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

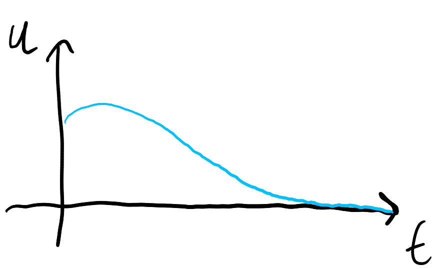
- Damping
  - Underdamped



complex roots

$c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$

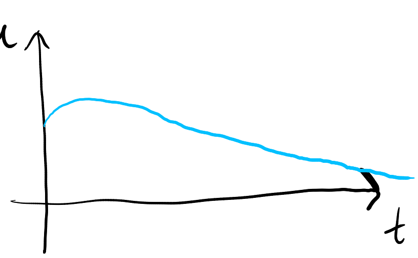
  - Critically damped



repeated roots

$c_1 e^{rt} + c_2 t e^{rt}$

  - Overdamped



distinct real roots

$c_1 e^{r_1 t} + c_2 e^{r_2 t}$

- To find the amplitude of a solution, use the following formula.

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta),$$

  where $R = \sqrt{A^2 + B^2}$ and $\delta = \arctan(B/A)$. The **amplitude** is R, and the **angular frequency** is ω.

- The **period** is $2\pi/$(angular frequency).
- **Natural frequency:** The frequency of the system when there is no damping and no external force.
- **Resonance** occurs when there is no damping and the external force of a system matches the natural frequency of the system. When this happens, the oscillations grow unboundedly large.

Exercise 1

Suppose there is a spring hanging from the ceiling. When we hang a 10 kg mass on the spring, it stretches by 50 cm. Find the natural frequency of the system. If we set the spring in motion by first stretching the spring an additional 10 cm and then pushing it upward with a velocity of 20 cm/s, find the amplitude, angular frequency, and period of the motion of the mass. (Use $g = 10\,\mathrm{m/s^2}$.)

$$k = \frac{mg}{L} = \frac{(10\text{ kg})(10\text{ m/s}^2)}{0.5\text{ m}} = 200\text{ N/m}$$

$$10u'' + 200u = 0, \quad u(0) = 0.1\text{ m}, \quad u'(0) = -0.2\text{ m/s}$$

$$10r^2 + 200 = 0$$

$$r^2 = -20$$

$$r = \pm i\sqrt{20}$$

$$u(t) = c_1\cos(\sqrt{20}t) + c_2\sin(\sqrt{20}t)$$

natural freq is $\sqrt{20}$ rad/s

$$u(0) = c_1 = 0.1$$

$$u'(t) = -\sqrt{20}c_1\sin(\sqrt{20}t) + \sqrt{20}c_2\cos(\sqrt{20}t)$$

$$u'(0) = \sqrt{20}c_2 = -0.2 \Rightarrow c_2 = \frac{-0.2}{\sqrt{20}}$$

$$u(t) = 0.1\cos(\sqrt{20}\,t) - \frac{0.2}{\sqrt{20}}\sin(\sqrt{20}\,t)$$

$$= \underbrace{\sqrt{(0.1)^2 + \left(\frac{0.2}{\sqrt{20}}\right)^2}}_{\text{amplitude (in m)}} \cos\left(\underbrace{\sqrt{20}}_{\text{angular freq (rad/s)}}\,t - \tan^{-1}\left(\frac{0.2}{0.1\sqrt{20}}\right)\right)$$

$$\text{period} = \frac{2\pi}{\sqrt{20}} \text{ sec}$$

Exercise 2

Suppose there is a 40 N mass hanging from a spring on the ceiling. The spring has a spring constant of 5 N/m. When the mass is moving 3 m/s, it experiences a damping force of 15 N. There is a external upward force of 9 N that turns on at time $t = 7\,\mathrm{sec}$. If the mass is started from equilibrium with an initial downward velocity of 2 m/s, write down the initial value problem that models this system. Is the system under, over, or critically damped?

$$m = \frac{40N}{10m/s^2} = 4kg \qquad \gamma = \frac{15N}{3m/s} = 5\frac{N}{m/s}$$

$$\boxed{4u'' + 5u' + 5u = 9u_7(t), \quad u(0) = 0m, \quad u'(0) = 2m/s}$$

$$4r^2 + 5r + 5 = 0$$

$$r = \frac{-5 \pm \sqrt{25 - 4(5)(4)}}{2(4)} = \frac{-5 \pm \sqrt{25-80}}{8}$$ complex roots

$\Rightarrow \boxed{\text{underdamped}}$

Exercise 3

Find the value of the damping coefficient γ in the previous example that would make the system critically damped.

$$4u'' + \gamma u' + 5u = 0$$

$$4r^2 + \gamma r + 5 = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(4)(5)}}{2(4)}$$

want repeated roots, so stuff under square root should be 0.

$$\gamma^2 - 80 = 0 \Rightarrow \boxed{\gamma = \sqrt{80}\ \frac{N}{m/s}}$$

LAPLACE TRANSFORMS

Review

- Definition of the Laplace transform

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t.$$

- General strategy for solving differential equations with the Laplace transform
  1. Laplace transform
  2. Solve for $Y(s)$
  3. Inverse transform
- Common Laplace transforms

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n $(n = 1, 2, ...)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n$ $(n = 1, 2, ...)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$u_c(t)$ $(c \geq 0)$	$\frac{e^{-cs}}{s}$	$s > 0$
$\delta(t-c)$ $(c \geq 0)$	e^{-cs}	

- Shift theorems

$$\begin{aligned} \mathcal{L}\{u_c(t)f(t-c)\} &= e^{-cs}F(s) \\ \mathcal{L}\{u_c(t)f(t)\} &= e^{-cs}\mathcal{L}\{f(t+c)\} \\ \mathcal{L}^{-1}\{e^{-cs}F(s)\} &= u_c(t)f(t-c) \\ \mathcal{L}^{-1}\{F(s-c)\} &= e^{ct}f(t) \end{aligned}$$

Exercise 4

Solve the initial value problem

$$f'' - 6f' + 9f = 0, \quad f(0) = 7, \quad f'(0) = 3.$$

$$s^2F(s) - sf(0) - f'(0) - 6\left(sF(s) - f(0)\right) + 9F(s) = 0$$

$$(s^2 - 6s + 9)F(s) - 7s - 3 + 42 = 0$$

$$F(s) = \frac{7s - 39}{(s-3)^2} = \frac{A}{(s-3)^2} + \frac{B}{s-3}$$

$$7s - 39 = A + B(s-3)$$

$$= \underbrace{Bs}_{=7} + \underbrace{A - 3B}_{=-39}$$

$$B = 7 \qquad A - 3B = -39$$

$$A = -39 + 3B$$
$$= -39 + 3(7)$$
$$= -39 + 21$$
$$= -18$$

$$F(s) = \frac{-18}{(s-3)^2} + \frac{7}{s-3}$$

$$\boxed{f(t) = -18\,te^{3t} + 7e^{3t}}$$

Exercise 5

Solve the initial value problem

$$y'''' + y'' = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 3, \quad y'''(0) = 0.$$

$$s^4 Y(s) - s^3 \cancelto{0}{y(0)} - s^2 \cancelto{0}{y'(0)} - s y''(0) - \cancelto{0}{y'''(0)} + s^2 Y(s) - s \cancelto{0}{y(0)} - \cancelto{0}{y'(0)} = 0$$

$$(s^4 + s^2) Y(s) - 3s = 0$$

$$Y(s) = \frac{3s}{s^2(s^2+1)} = \frac{3}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1}$$

$$3 = A(s^2+1) + (Bs+C)s$$

$$= As^2 + A + Bs^2 + Cs$$

$$= \underbrace{(A+B)}_{=0}s^2 + \underbrace{C}_{=0}s + \underbrace{A}_{=3}$$

$$B = -A = -3 \qquad C = 0 \qquad A = 3$$

$$Y(s) = \frac{3}{s} - \frac{3s}{s^2+1}$$

$$\boxed{y(t) = 3 - 3\cos(t)}$$

Exercise 6

Solve the initial value problem

$$y'' - 9y = u_2(t), \quad y(0) = 0, \quad y'(0) = 0.$$

$$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} - 9Y(s) = \frac{e^{-2s}}{s}$$

$$(s^2 - 9) Y(s) = \frac{e^{-2s}}{s}$$

$$Y(s) = e^{-2s} \frac{1}{s(s+3)(s-3)} = e^{-2s}\left(\frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-3}\right)$$

$$1 = A(s+3)(s-3) + Bs(s-3) + Cs(s+3)$$

$s=0$: $1 = -9A \Rightarrow A = -1/9$

$s=-3$: $1 = 18B \Rightarrow B = 1/18$

$s=3$: $1 = 18C \Rightarrow C = 1/18$

$$Y(s) = e^{-2s}\left(\frac{-1}{9}\frac{1}{s} + \frac{1}{18}\frac{1}{s+3} + \frac{1}{18}\frac{1}{s-3}\right)$$

$$\boxed{y(t) = \frac{-1}{9}u_2(t) + \frac{1}{18}u_2(t)e^{-3(t-2)} + \frac{1}{18}u_2(t)e^{3(t-2)}}$$

Exercise 7

Convert the following piecewise function into a function that involves step functions.

$$f(t) = \begin{cases} t & t < 3 \\ t^2 & 3 \leq t < 5 \\ t^3 & t \geq 5 \end{cases}$$

$$f(t) = t - t u_3(t) + t^2 u_3(t) - t^2 u_5(t) + t^3 u_5(t)$$

Exercise 8

Find the inverse transform of $H(s) = \dfrac{5e^{-3s}s^2}{(s+1)(s^2+4)}$. $= e^{-3s}\left(\dfrac{5s^2}{(s+1)(s^2+4)}\right)$

$$= e^{-3s}\left(\frac{A}{s+1} + \frac{Bs+C}{s^2+4}\right)$$

$$5s^2 = A(s^2+4) + (Bs+C)(s+1)$$

$$= As^2 + 4A + Bs^2 + Cs + Bs + C$$

$$= \underbrace{(A+B)}_{=5}s^2 + \underbrace{(B+C)}_{=0}s + \underbrace{4A+C}_{=0}$$

$A+B=5$ $\quad B=-C \quad A = -\frac{1}{4}C$

$-\frac{1}{4}C - C = 5 \quad B = 4 \quad A = -\frac{1}{4}(-4) = 1$

$-\frac{5}{4}C = 5$

$C = -4$

$$H(s) = e^{-3s}\left(\frac{1}{s+1} + \frac{4s-4}{s^2+4}\right) = e^{-3s}\left(\frac{1}{s+1} + 4\frac{s}{s^2+4} - 2\frac{2}{s^2+4}\right)$$

$$\boxed{h(t) = e^{-(t-3)} + 4\cos(2(t-3)) - 2\sin(2(t-3))}$$

Exercise 9

Convert the following function into piecewise form.

$$g(t) = u_3(t)\cos(t) - tu_6(t).$$

$$g(t) = \begin{cases} 0 & t < 3 \\ \cos(t) & 3 \le t < 6 \\ \cos(t) - t & t \ge 6 \end{cases}$$

Exercise 10

Find the Laplace transform of $f(t) = \cos(t-2)u_2(t) + \delta(t-5) + \int_0^t \tau \cos(2t - 2\tau)\, \mathrm{d}\tau$.

$$F(s) = e^{-2s}\mathcal{L}\{\cos(t+2-2)\} + e^{-5s} + \mathcal{L}\{t\}\mathcal{L}\{\cos(2t)\}$$

$$= e^{-2s}\mathcal{L}\{\cos(t)\} + e^{-5s} + \frac{1}{s^2}\frac{s}{s^2+4}$$

$$\boxed{= e^{-2s}\frac{s}{s^2+1} + e^{-5s} + \frac{s}{s^2(s^2+4)}}$$

Exercise 11

Using the definition of the Laplace transform, show that $\mathcal{L}\{e^{3t}\} = \dfrac{1}{s-3}$.

$$\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{-st} e^{3t}\, dt$$

$$= \int_0^\infty e^{t(3-s)}\, dt$$

$$= \frac{1}{3-s} e^{(3-s)t} \Big|_{t=0}^{\infty}$$

$$= \frac{1}{3-s}\Big(\underbrace{\lim_{t\to\infty} e^{(3-s)t}}_{0 \text{ if } s>3} - 1\Big)$$

$$= \frac{-1}{3-s}$$

$$\boxed{= \frac{1}{s-3} \quad \text{for } s>3}$$

SERIES SOLUTIONS TO DIFFERENTIAL EQUATIONS

Review

- A power series centered at x_0 has the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

- Basic strategy for solving differential equations with power series
  1. Plug the power series into the differential equation
  2. Combine the sums
     (a) Bring any x's inside
     (b) Make the powers match (by shifting the index)
     (c) Make the lower bound match (by peeling off terms)
  3. Find the recurrence relation
  4. Solve for the first few a_n in terms of a_0 and a_1
  5. Factor out a_0 and a_1 to find y_1 and y_2
- If you have initial conditions, then $y(x_0) = a_0$ and $y'(x_0) = a_1$, where x_0 is the center of the power series.

Exercise 12

Find a series solution centered at $x_0 = 0$ to the differential equation. Find y_1 and y_2 up to the x^4 term.

$$(x+1)f'' + f = 0.$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(x+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \underline{\qquad\qquad\qquad\qquad} = 0$$

$n \to n+1$ $\qquad$ $n \to n+2$

$$\sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underline{\qquad\qquad} + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$\underbrace{a_0 + 2a_2}_{=0} + \sum_{n=1}^{\infty} \underbrace{\left[(n+1)n\, a_{n+1} + (n+2)(n+1) a_{n+2} + a_n\right]}_{=0} x^n = 0$$

$$a_2 = \frac{-1}{2} a_0 \qquad a_{n+2} = \frac{-(n+1)n\, a_{n+1} - a_n}{(n+2)(n+1)} \qquad n = 1, 2, 3, \dots$$

$$a_3 = \frac{-2a_2 - a_1}{3 \cdot 2} = -\frac{1}{3}a_2 - \frac{1}{6}a_1 = -\frac{1}{3}\left(\frac{-1}{2}a_0\right) - \frac{1}{6}a_1$$

$$= \frac{1}{6}a_0 - \frac{1}{6}a_1$$

$$a_4 = \frac{-3 \cdot 2 a_3 - a_2}{4 \cdot 3} = -\frac{1}{2}a_3 - \frac{1}{12}a_2 = -\frac{1}{2}\left(\frac{1}{6}a_0 - \frac{1}{6}a_1\right) - \frac{1}{12}\left(\frac{-1}{2}a_0\right)$$

$$= \frac{-1}{12}a_0 + \frac{1}{12}a_1 + \frac{1}{24}a_0 = \frac{-1}{24}a_0 + \frac{1}{12}a_1$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{1}{2}a_0 x^2 + \left(\frac{1}{6}a_0 - \frac{1}{6}a_1\right)x^3 + \left(\frac{-1}{24}a_0 + \frac{1}{12}a_1\right)x^4 + \cdots$$

$$= a_0 \underbrace{\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots\right)}_{y_1(x)}$$

$$+ a_1 \underbrace{\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \cdots\right)}_{y_2(x)}$$

Exercise 13

Start finding a series solution centered at $x_0 = 1$ to the differential equation. Stop once you find the recurrence relation.

$$y'' - xy' + 3y = 0.$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + 3 \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\underline{\qquad\qquad} - (x-1+1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \underline{\qquad\qquad} = 0$$

$$\underline{\qquad\qquad} - (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \underline{\qquad\qquad} = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}}_{n \to n+2} - \sum_{n=1}^{\infty} n a_n (x-1)^{n} - \underbrace{\sum_{n=1}^{\infty} n a_n (x-1)^{n-1}}_{n \to n+1} + 3 \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n + 3 \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - a_1 - \sum_{n=1}^{\infty} (n+1) a_{n+1} (x-1)^n + 3a_0 + 3 \sum_{n=1}^{\infty} a_n (x-1)^n = 0$$

$$\underbrace{3a_0 - a_1 + 2a_2}_{=0} + \sum_{n=1}^{\infty} \underbrace{\left[(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} + 3a_n \right]}_{=0} (x-1)^n = 0$$

$$\boxed{a_2 = \frac{a_1 - 3a_0}{2} \qquad a_{n+2} = \frac{(n+1) a_{n+1} + n a_n - 3a_n}{(n+2)(n+1)} \quad \text{for } n = 1, 2, 3, \ldots}$$

Exercise 14

Find the first 4 terms of the solution $y(x) = \sum_{n=0}^{\infty} a_n(x+5)^n$ if the recurrence relation is

$$a_{n+2} = \frac{2a_{n+1} - a_n}{n+1}, \qquad \text{for } n = 0, 1, 2, \dots$$

and the initial conditions are $\underbrace{y(-5) = 1}_{a_0=1}$ and $\underbrace{y'(-5) = -1}_{a_1=-1}$.

$$a_2 = \frac{2a_1 - a_0}{1} = 2(-1) - 1 = -3$$

$$a_3 = \frac{2a_2 - a_1}{2} = a_2 - \frac{1}{2}a_1 = -3 - \frac{1}{2}(-1) = -3 + \frac{1}{2} = -\frac{5}{2}$$

$$y(x) = a_0 + a_1(x+5) + a_2(x+5)^2 + a_3(x+5)^3 + \dots$$

$$\boxed{= 1 - (x+5) - 3(x+5)^2 - \frac{5}{2}(x+5)^3 + \dots}$$