7.4: BASIC THEORY OF SYSTEMS OF 1ST-ORDER LINEAR EQUATIONS

Review

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

- **Existence and uniqueness**: Consider the initial value problem

  and I contains t_0

  $$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

  If all the entries of $P(t)$ and $g(t)$ are continuous functions on an open interval $I = (a, b)$, then there exists a unique solution to the initial value problem on the interval I.

- **Principle of superposition**: If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions to the differential equation $\mathbf{x}' = P(t)\mathbf{x}$, then

  $$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

  is also a solution.

- **Wronskian for vector functions**: If $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ are all n-vectors, then their Wronskian is defined as

  $$W[\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)}](t) = \det \mathbf{X}(t), \qquad X(t) = \left[\vec{x}^{(1)} \,|\, \cdots \,|\, \vec{x}^{(n)}\right]$$

  where $\mathbf{X}(t)$ is the matrix whose columns are $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$.

- **Fundamental set of solutions:** Suppose $P(t)$ is an $n \times n$ matrix. Then, $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ is a fundamental set of solutions if their Wronskian is nonzero.

- **General solution**

  $\{\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}\}$ are a fundamental set of solutions $\iff$ $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)}$ is the general solution

- **Abel's theorem**: If $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ are solutions to $\mathbf{x}' = P(t)\mathbf{x}$ on an interval I, then their Wronskian is either always zero or never zero on I.

  Practical consequence: You only need to check the Wronskian at a single point in the interval where the solution exists.

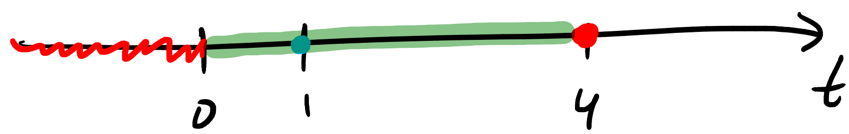
Exercise 1

Where is the following initial value problem guaranteed to have a unique solution?

$$\mathbf{x}' = \begin{bmatrix} 3 & -t^2+2 \\ \ln(t) & \cos(t) \end{bmatrix}\mathbf{x} + \begin{bmatrix} (t-4)^{-3} \\ 14e^t \end{bmatrix}, \qquad \mathbf{x}(1) = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

$\ln(t)$ is not defined if $t \leq 0$.

$\frac{1}{(t-4)^3} = (t-4)^{-3}$ is not defined at $t = 4$



There is a unique solution on $(0,4)$.

Exercise 2

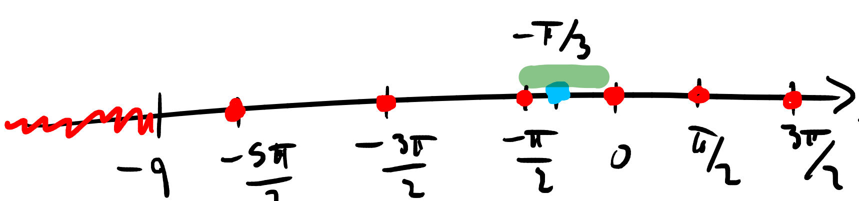
Where is the following initial value problem guaranteed to have a unique solution?

$$\mathbf{x}' = \begin{bmatrix} \tan(t) & \pi \\ \frac{3}{t} & 7t \end{bmatrix}\mathbf{x} + \begin{bmatrix} 8 \\ \sqrt{t+9} \end{bmatrix}, \qquad \mathbf{x}(-\pi/3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\tan(t)$ is not continuous at $t = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$3/t$ is not continuous at $t = 0$.

$\sqrt{t+9} \Rightarrow$ need $t+9 \geq 0 \Rightarrow t \geq -9$



$\boxed{\text{unique solution on } \left(-\frac{\pi}{2}, 0\right)}$

Exercise 3

Consider the system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \mathbf{x}.$$

Is the following a fundamental set of solutions?

$$\left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t}, \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{t} \right\}$$

$\begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t}$:

$$8\begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t}$$

$$\begin{bmatrix} 40 \\ 16 \end{bmatrix} e^{8t} = \begin{bmatrix} 50-10 \\ 40-24 \end{bmatrix} e^{8t} = \begin{bmatrix} 40 \\ 16 \end{bmatrix} e^{8t} \checkmark$$

$\begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{t}$:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{t} = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{t} = \begin{bmatrix} 40-10 \\ 32-24 \end{bmatrix} e^{t} = \begin{bmatrix} 30 \\ 8 \end{bmatrix} e^{t} \times$$

this is not a solution.

So, no, they are not a fundamental set of solutions.

Exercise 4

Consider the system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 1/2 & 0 \\ 1 & -1/2 \end{bmatrix} \mathbf{x}.$$

Is the following the general solution?

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2}$:

$$\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2} = \begin{bmatrix} 1/2 & 0 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} e^{t/2}$$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2}$:

$$-\frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2} = \begin{bmatrix} 1/2 & 0 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} e^{-t/2}$$

Check the Wronskian:

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \det\left(\begin{bmatrix} \frac{1}{2}e^{t/2} & 0 \\ \frac{1}{2}e^{t/2} & -\frac{1}{2}e^{-t/2} \end{bmatrix}\right)$$

At $t=0$: $\det\left(\begin{bmatrix} 1/2 & 0 \\ 1/2 & -1/2 \end{bmatrix}\right) = \frac{-1}{4} \neq 0$

So, this is the general solution.

7.5: HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

Review

- How to solve a homogeneous linear system with constant coefficients (when you have distinct real eigenvalues)
  1. Assume your solution has the form $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$.
  2. Plug this in to get an eigenvalue problem.
  3. Solve for the eigenvalues r_1 and r_2 and the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
  4. The general solution is $c_1\boldsymbol{\xi}^{(1)}e^{r_1t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2t}$.
- **Phase plane/portrait**: A phase plane/portrait is essentially a 2D version of the phase line. It shows you where the solution moves as time passes.
- An **equilibrium point** is a point where if you start there, you will remain there forever. The origin is always an equilibrium point of the differential equation system $\mathbf{x}' = A\mathbf{x}$.
- **Stability** of equilibrium points
  - **Asymptotically stable**: If you start near the equilibrium point, you will be sucked into it as $t \to \infty$.
  - **Stable**: If you start near the equilibrium point, you will stay near it.
  - **Unstable**: There is at least one point near the equilibrium point that goes away from the equilibrium point.

Exercise 5

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \mathbf{x}$$

$$\vec{x}(t) = \vec{\xi} e^{rt}$$

$$r \vec{\xi} \cancel{e^{rt}} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{\xi} \cancel{e^{rt}}$$

$\Rightarrow$ r is an eigenvalue with corresponding eigenvector $\vec{\xi}$.

characteristic eq: $r^2 - \mathrm{tr}(A) r + \det(A) = 0$.

$$r^2 - 4r - 5 = 0$$

$$(r-5)(r+1) = 0$$

$$r = -1, 5 \leftarrow \text{eigenvalues}$$

eigenvector corresponding to $r = -1$:

$$\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow 2\xi_1 + 2\xi_2 = 0 \Rightarrow \xi_1 = -\xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -\xi_2 \\ \xi_2 \end{bmatrix} = \overset{1}{\cancel{\xi_2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \xi_1 \\ -\xi_1 \end{bmatrix} = \xi_1$$

eigenvector for $r=5$:

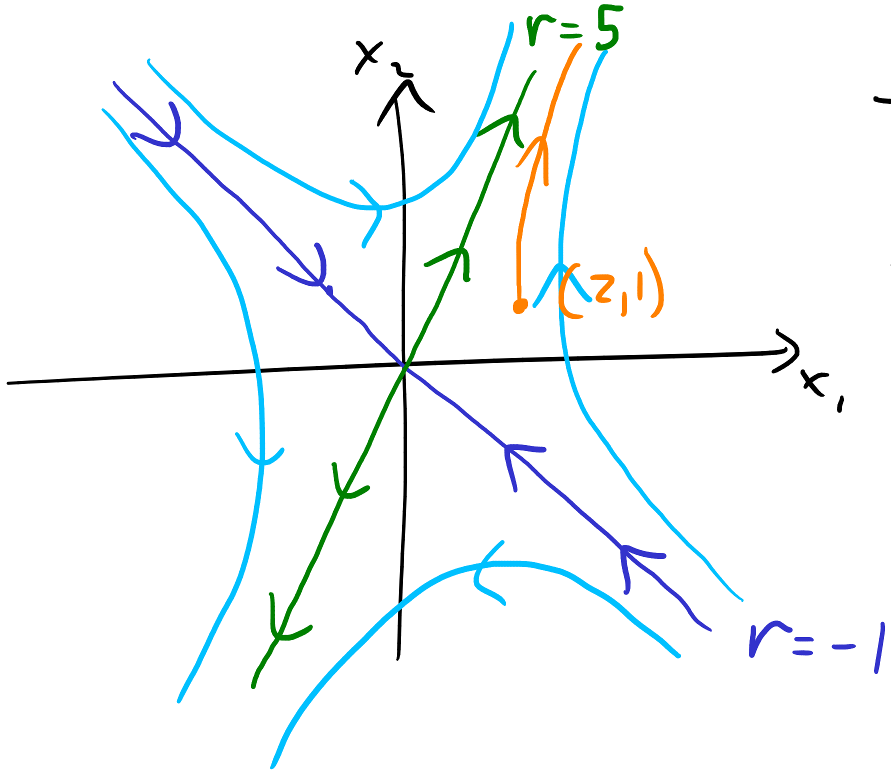
$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \vec{\xi} = \vec{0} \implies -4\xi_1 + 2\xi_2 = 0 \implies \xi_2 = 2\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ 2\xi_1 \end{bmatrix} = \overset{1}{\cancel{\xi_1}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

ξ = ksi = xi

general solution:

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}}$$



The origin is an unstable saddle point.

Solve the initial value problem when $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Draw this solution on the phase plane and sketch the graph of $x_1(t)$ and $x_2(t)$.

$$\vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0$$

$$= \begin{bmatrix} -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 2c_2 \end{bmatrix}$$

$$= \begin{bmatrix} -c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$-c_1 + c_2 = 2 \Rightarrow c_2 = 2 + c_1 = 2 - 1 = 1$$

$$c_1 + 2c_2 = 1 \Rightarrow c_1 + 2(2 + c_1) = 1$$

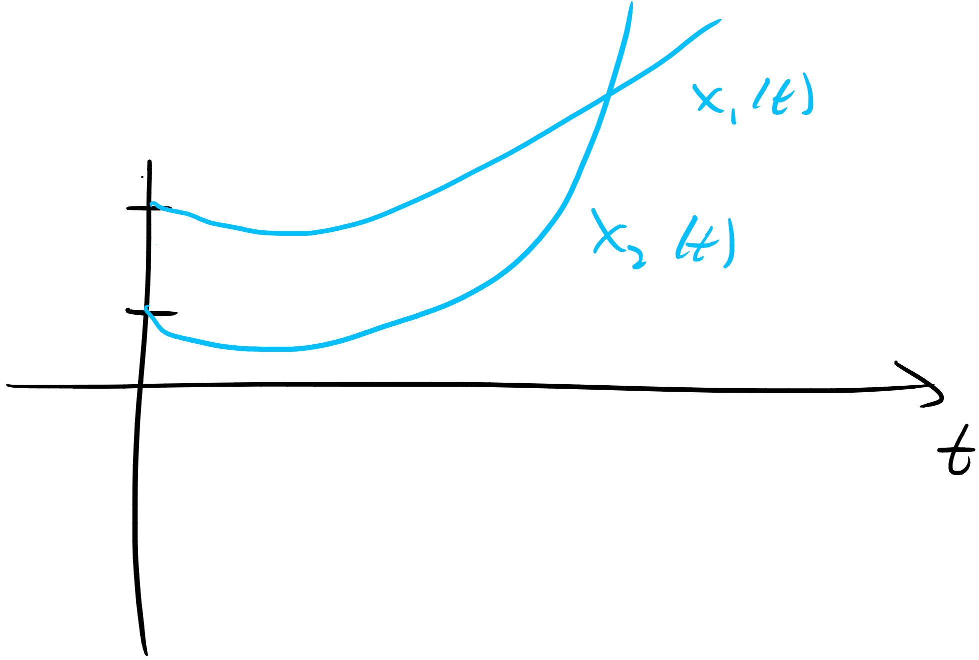
$$c_1 + 4 + 2c_1 = 1$$

$$3c_1 = -3$$

$$c_1 = -1$$

$$\boxed{\vec{x}(t) = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}}$$

$$= \begin{bmatrix} e^{-t} + e^{5t} \\ -e^{-t} + 2e^{5t} \end{bmatrix} \begin{matrix} \leftarrow x_1(t) \\ \leftarrow x_2(t) \end{matrix}$$



$x_1(t)$
$x_2(t)$
t

Exercise 6

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x_1' = -5x_1 + 4x_2$$
$$x_2' = \frac{3}{2}x_1 - 4x_2$$

$$\vec{x}\,' = \begin{bmatrix} -5 & 4 \\ 3/2 & -4 \end{bmatrix} \vec{x}$$

Characteristic eq: $r^2 + 9r + 14 = 0$

$(r+2)(r+7) = 0$

$r = -2, -7.$ ← eigenvalues

eigenvector for $r=-2$:

$$\begin{bmatrix} -3 & 4 \\ 3/2 & -2 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow -3\xi_1 + 4\xi_2 = 0$$
$$\xi_2 = \frac{3}{4}\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \frac{3}{4}\xi_1 \end{bmatrix} = \overset{4}{\cancel{\xi_1}} \begin{bmatrix} 1 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

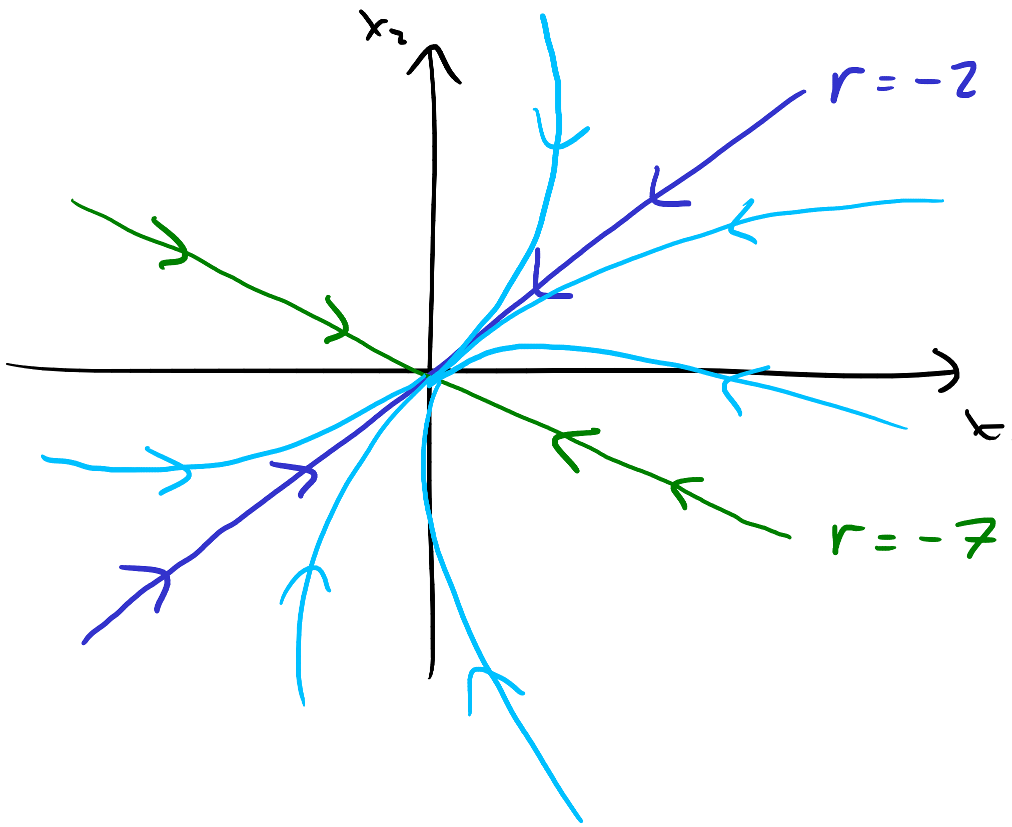
eigenvector for $r=-7$:

$$\begin{bmatrix} 2 & 4 \\ 3/2 & 3 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow 2\xi_1 + 4\xi_2 = 0 \Rightarrow \xi_2 = -\frac{1}{2}\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ -2\xi_1 \end{bmatrix} = \overset{2}{\cancel{\xi_1}} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

general solution:

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix}4\\3\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}2\\-1\end{bmatrix}e^{-7t}}$$



The origin is an asymptotically stable node.

Exercise 7

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x' = 2x + 2y$$
$$y' = x + 3y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{x}\,' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$$

Characteristic eq: $r^2 - 5r + 4 = 0$

$$(r-1)(r-4) = 0$$

$$r = 1, 4$$

eigenvector for $r=1$:

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \quad \xi_1 + 2\xi_2 = 0 \Rightarrow \xi_1 = -2\xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -2\xi_2 \\ \xi_2 \end{bmatrix} = \xi_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

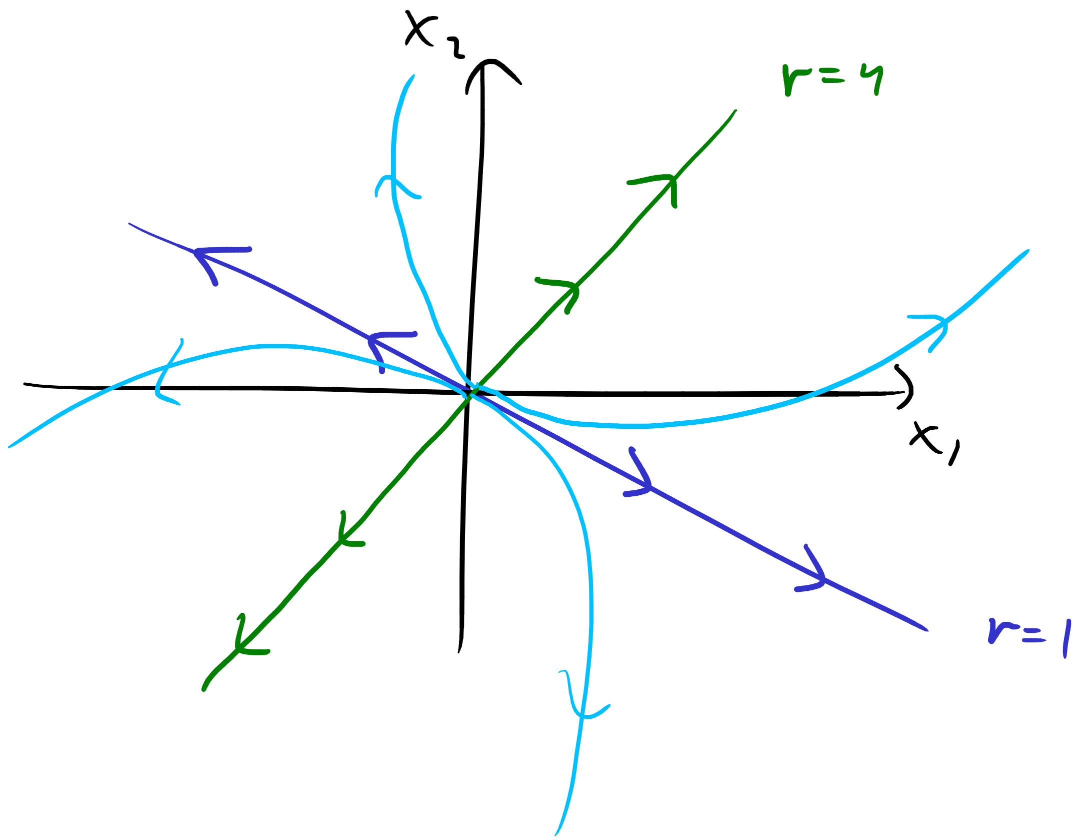
eigenvector for $r=4$:

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \quad \xi_1 - \xi_2 = 0 \Rightarrow \xi_1 = \xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \xi_2 \end{bmatrix} = \xi_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

general solution:

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}}$$



The origin is an unstable node.

7.6: COMPLEX EIGENVALUES

Review

- To solve the system $\mathbf{x}' = A\mathbf{x}$ when you have complex eigenvectors:
  - Solve for just **one** of the eigenvectors.
  - Separate $\boldsymbol{\xi} e^{rt}$ into its real and imaginary parts.
  - The real and imaginary parts form a fundamental set of solutions.
    - (Assuming that A is 2×2. If A is larger, than there are also more solutions.)

Exercise 8

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x' = 3x + y$$
$$y' = -2x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \vec{x}$$

characteristic eq: $r^2 - 4r + 5 = 0$

$$r = \frac{4 \pm \sqrt{16 - 4(5)}}{2} = 2 \pm \frac{\sqrt{-4}}{2} = 2 \pm \frac{2i}{2} = 2 \pm i$$

eigenvector corresponding to $2 - i$:

$$\left(\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} - (2-i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 3-(2-i) & 1 \\ -2 & 1-(2-i) \end{bmatrix} \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow (1+i)\xi_1 + \xi_2 = 0$$

$$\xi_2 = -(1+i)\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ -(1+i)\xi_1 \end{bmatrix} = \cancelto{1}{\xi_1} \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

So, $\begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{(2-i)t}$ is a solution.

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$= \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{2t-it} = \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{2t} e^{-it}$$

$$= \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{2t} \left(\underbrace{\cos(-t)}_{\cos(t)} + i \underbrace{\sin(-t)}_{-\sin(t)} \right)$$

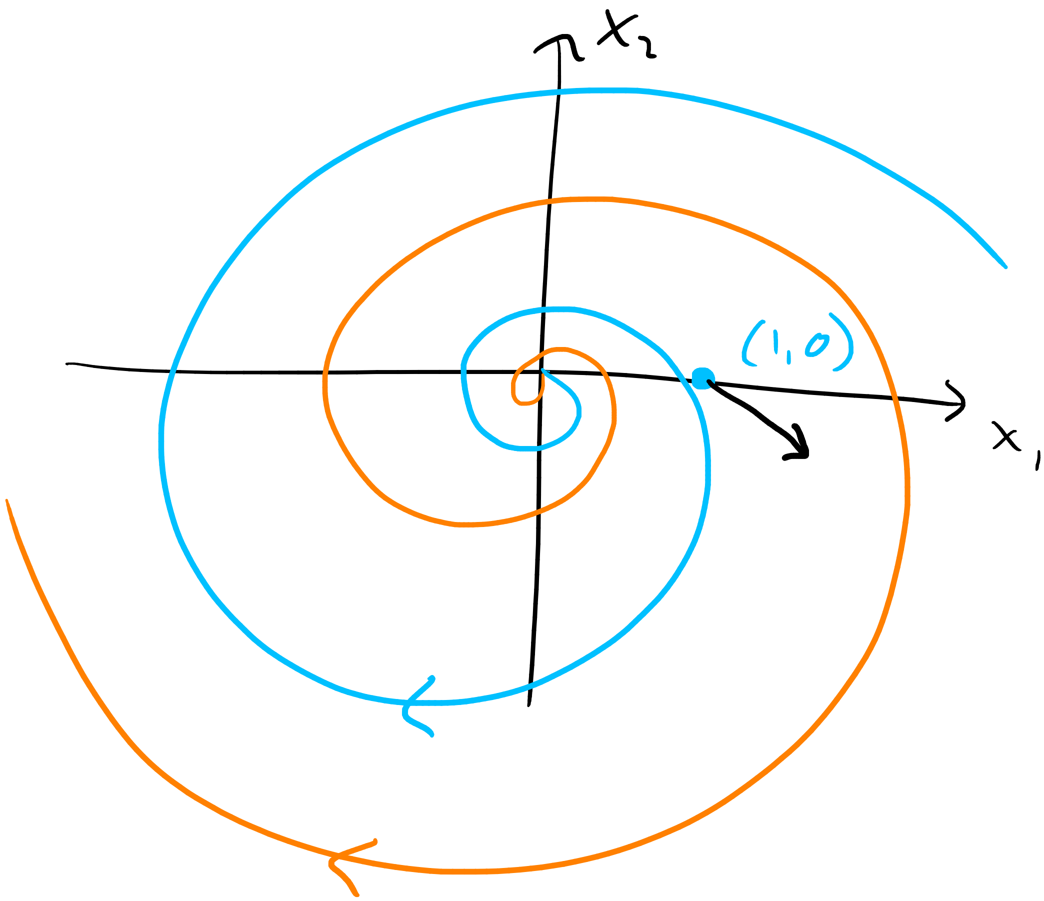
$$= \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{2t} \left(\cos(t) - i\sin(t) \right)$$

$$= e^{2t} \begin{bmatrix} \cos(t) - i\sin(t) \\ -\cos(t) + i\sin(t) - i\cos(t) + \cancelto{-1}{i^2}\sin(t) \end{bmatrix}$$

$$= \underbrace{e^{2t} \begin{bmatrix} \cos(t) \\ -\cos(t) - \sin(t) \end{bmatrix}}_{\vec{x}^{(1)}} + i \underbrace{e^{2t} \begin{bmatrix} -\sin(t) \\ \sin(t) - \cos(t) \end{bmatrix}}_{\vec{x}^{(2)}}$$

general solution:

$$\boxed{\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} \cos(t) \\ -\cos(t) - \sin(t) \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -\sin(t) \\ \sin(t) - \cos(t) \end{bmatrix}}$$



$$\vec{x}' = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

The origin is an unstable spiral point.

Exercise 9

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

characteristic eq: $r^2 - 0r + 4 = 0$

$$r^2 = -4$$

$$r = \pm 2i$$

eigenvector for $r = -2i$:

$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \vec{\xi} = \vec{0} \implies \begin{array}{l} 2i\xi_1 - 2\xi_2 = 0 \\ \xi_2 = i\xi_1 \end{array}$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ i\xi_1 \end{bmatrix} = \overset{1}{\cancel{\xi_1}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So, $\begin{bmatrix} 1 \\ i \end{bmatrix} e^{-2it}$ is a solution.

$$= \begin{bmatrix} 1 \\ i \end{bmatrix} \left(\underbrace{\cos(-2t)}_{\cos(2t)} + i \underbrace{\sin(-2t)}_{-\sin(2t)} \right)$$

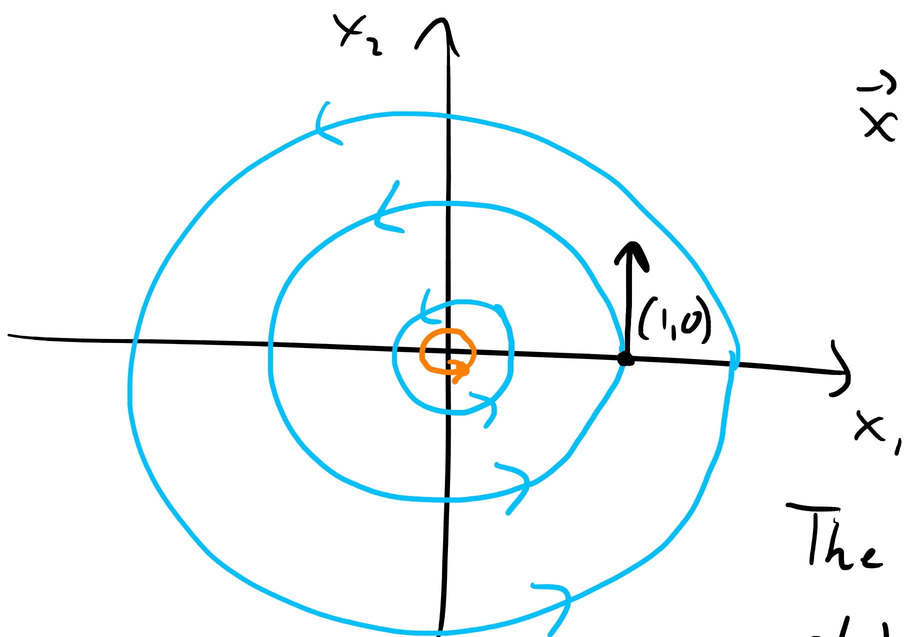
$$= \begin{bmatrix} 1 \\ i \end{bmatrix} \left(\cos(2t) - i\sin(2t) \right)$$

$$= \begin{bmatrix} \cos(2t) - i\sin(2t) \\ i\cos(2t) - \cancel{i^2}^{\,-1}\sin(2t) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}}_{\vec{x}^{(1)}} + i \underbrace{\begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}}_{\vec{x}^{(2)}}$$

general solution:

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}}$$



$$\vec{x}\,' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The origin is a stable center.

Exercise 10

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin. Solve the initial value problem with $\mathbf{x}(0) = [-1 \;\; 2]^T$.

$$\mathbf{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \mathbf{x}$$

Characteristic eq: $r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm \frac{\sqrt{-16}}{2} = -1 \pm \frac{4i}{2} = -1 \pm 2i$$

eigenvector for $-1+2i$:

$$\begin{bmatrix} 1-(-1+2i) & -8 \\ 1 & -3-(-1+2i) \end{bmatrix} \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 2-2i & -8 \\ 1 & -2-2i \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \begin{aligned} \xi_1 + (-2-2i)\xi_2 &= 0 \\ \xi_1 &= (2+2i)\xi_2 \end{aligned}$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} (2+2i)\xi_2 \\ \xi_2 \end{bmatrix} = \cancel{\xi_2} \, 1 \begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} 2+2i \\ 1 \end{bmatrix} e^{(-1+2i)t}$ is a solution.

$$= \begin{bmatrix} 2+2i \\ 1 \end{bmatrix} e^{-t} e^{2it}$$

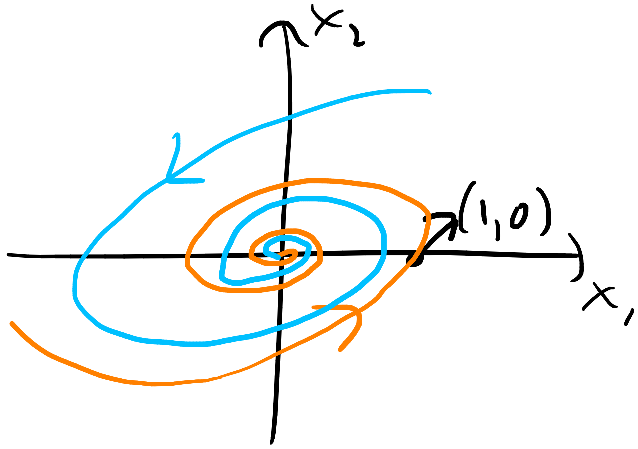
$$= \begin{bmatrix} 2+2i \\ 1 \end{bmatrix} e^{-t}\left(\cos(2t) + i\sin(2t)\right)$$

$$= e^{-t}\begin{bmatrix} 2\cos(2t) + 2i\sin(2t) + 2i\cos(2t) + 2\overset{-1}{\cancel{i^2}}\sin(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

$$= \underbrace{e^{-t}\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix}}_{\vec{x}^{(1)}} + i\underbrace{e^{-t}\begin{bmatrix} 2\sin(2t) + 2\cos(2t) \\ \sin(2t) \end{bmatrix}}_{\vec{x}^{(2)}}$$

general solution:

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\sin(2t) + 2\cos(2t) \\ \sin(2t) \end{bmatrix}$$



$$\vec{x}\,' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The origin is an asymptotically stable spiral point.

$$\vec{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 + 2c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$c_1 = 2$$

$$2c_1 + 2c_2 = -1$$

$$4 + 2c_2 = -1$$

$$2c_2 = -5$$

$$c_2 = -5/2$$

$$\boxed{\vec{x}(t) = 2e^{-t} \begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix} - \frac{5}{2} e^{-t} \begin{bmatrix} 2\sin(2t) + 2\cos(2t) \\ \sin(2t) \end{bmatrix}}$$